

# Stochastic Volatility and Mean-variance Analysis

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## 1 Introduction

Stochastic volatility models usually lead to a linear option pricing equation containing a market price of risk term. This term is the source of endless problems and argument.

The main reason for the argument is that this quantity is not directly observable. At best it can be deduced from the prices of derivatives, so called 'fitting.' But this is far from adequate, the fitting will only work if those who set the prices of derivatives are using the same model and they are consistent in that the fitted market price of risk does not change when the model is refitted a few days later.

In practice, refitted parameters are always significantly different from the original fit. This is why practitioners use static hedging, to minimize model error. However, static hedging may be considered to be an afterthought, since it is, in the classical framework, no more than a patch for mending a far-from-perfect model.

Whether we have a deterministic volatility surface or a stochastic volatility model with prescribed or fitted market price of risk, we will always be faced with how to interpret refitting. Was the market wrong before but is now right, or was the market correct initially and now there are arbitrage opportunities? We won't be faced with awkward questions like this if we don't expect our model, whatever it may be, to give unique values. In this paper we'll see how to estimate probabilities for prices being correct. We do this by only delta hedging and not dynamically vega hedging. Instead we look at means and variances for option values.

## 2 What's Wrong

In the mark-to-market accounting framework, the price of a security should be marked at the prevailing market price. Thus, we do not need a theoretical model to price vanilla products in this framework. A model plays a significant role for determining the price of custom products as well as for risk management. A typical approach in practice is to select a

suitable model and to calibrate its parameter to match the model price of a vanilla product with the market quote:

$$\begin{aligned} \text{model::price}(\text{product}(\alpha), \text{quote}(\text{product}(\cdot), t), \text{parameter}, t) \\ = \text{quote}(\text{product}(\alpha), t). \end{aligned}$$

whenever a quote is available. Thus, by the implicit function theorem, the model parameter is a function of market prices and time:

$$\text{parameter} = \theta(\text{quote}(\text{product}(\cdot), t), t).$$

This function is *supposed to be invariant* under the change of time and quote, but there is no physical constraint that it *has to be invariant*. A model uses its parameter to describe the random behavior of the market prices. Therefore, if the model parameter changes, the prices before and after the change are not consistent any more. This will generate P&L that is unexplained by the model.

In the stochastic volatility framework, it looks as if the model allows its parameter to change. However, the volatility in this context is not a parameter any more. It is just an index which is assumed observable or estimable. The parameter is one that describes the dynamics of the volatility. The obvious merit of a stochastic volatility model is that it has more parameters to fit the market quotes better (for example, the smile). Nevertheless, its parameter is not immune from changing randomly in time. This is simply because the market itself has a higher order of complexity than that of a stochastic volatility model.

Suppose that a stochastic volatility model has its parameters invariant under the change of time and quote. Does this mean that this model leads us to a risk-free land? Here's an extreme example. The market is pricing all the vanilla options with a flat volatility at 50%. One calibrates the Black-Scholes model perfectly, always. What happens if the realized volatility of the underlying price won't agree with 50%? Thus, a perfect and stable calibration does not necessarily immunize the portfolio.

Market prices are subject to supply and demand. Since the buy-side and sell-side may have different rules (such as a short selling constraint) and asymmetric information, there is a possibility of price elevation, also known as a bubble. The typical approach, where model parameters are fitted to match the market prices, will not help you to manage the risk better in such a case.

There are other problems. A significant one in practice is that the meaning of vega hedging is ambiguous. One interpretation is that it is a hedge against the change of the portfolio value with respect to the change of implied volatilities (i.e. market prices of vanilla products). In this case, one has to re-calibrate the model by bumping the market prices to obtain the sensitivity. Another interpretation is that it is a hedge against the change of the portfolio value with respect to the change of volatility index, that is assumed observable but never is. In this case, the sensitivity is obtained from the model without necessarily bumping the market prices. The first one complies with the motivation of the mark-to-market framework. The second is more faithful to theory. Neither one is perfect. When these two are different, we are in serious trouble, as a wrong choice will give a mishedge.

### 3 The Model for the Asset and its Volatility

We are going to work with a general stochastic volatility model

$$dS(t) = \mu(S(t), Z(t), t)S(t) dt + \sigma(S(t), Z(t), t)S(t) dX_1(t)$$

and

$$dZ(t) = p(S(t), Z(t), t) dt + q(S(t), Z(t), t) dX_2(t)$$

where  $X_1$  and  $X_2$  are standard Brownian motions under physical measure with an instantaneous correlation  $d[X_1, X_2](t) = \rho(S(t), Z(t), t)dt$ . If the coefficient function  $\sigma(s, z, t) = z$ , the above specification agrees with the classical setting. We'll only consider a non-dividend-paying asset, the modifications needed to allow for dividends are the usual. In what follows we will drop the time index  $(t)$  and function arguments  $(S(t), Z(t), t)$  as long as the expressions are clear.

We are going to examine the statistical properties of a portfolio that tries to replicate as closely as possible the original option position. We will not hedge the portfolio dynamically with other options so our portfolio will not be risk free. Instead we will examine the mean and variance of the value of our portfolio as it varies through time.

With  $\Pi$  representing the discounted cash-flow of maintaining  $-\Delta$  in the asset dynamically:

$$\Pi(t, T) = \sum_{i=1}^n e^{-r(\tau_i - t)} P(S(\tau_i), \tau_i) - \int_t^T e^{-r(\tau - t)} \Delta (dS(\tau) - rS(\tau)d\tau)$$

where  $P$  denotes a payoff of a contingent claim at  $\tau_i \in [t, T]$ , which can be a stopping time, and where  $r$  denotes a funding cost rate. One can make  $r$  time dependent, but we'll keep things simple here. Except those times when a claim is settled, the change in this cash-flow in time is continuous:

$$\Pi(t, T) = (1 - rdt)\Pi(t + dt, T) - \Delta(dS(t) - rS(t)dt). \quad (1)$$

We are going to vary  $\Delta$  dynamically so as to replicate as closely as possible the option payoff. At expiration we will hold stock, and have a cash account containing the results of our trading. We are going to analyze the mean and the variance of our total position and interpret this in terms of option prices and probabilities.

### 4 Analysis of the Mean

Naturally we are to determine the trading strategy  $\Delta$  in a Markovian way. In fact, the stochastic control problem is reduced to a Markov control problem under a mild regularity condition, and therefore we will simply start from this for now. Define the mean (or the expected future cash flow)  $m$  at any time by

$$m(S(t), Z(t), t) = E_t[\Pi(t, T)]$$

where the expectation  $E_t$  is a shorthand notation for the conditional expectation given the state of the world at time  $t$ . Using the equation (1), we obtain:

$$m = E_t[(1 - rdt)(m + dm) - \Delta(dS(t) - rS(t)dt)].$$

Thus, using Itô's formula we obtain the following partial differential equation (PDE):

$$\begin{aligned} \frac{\partial m}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 m}{\partial S^2} + \rho\sigma Sq \frac{\partial^2 m}{\partial S \partial Z} + \frac{1}{2}q^2 \frac{\partial^2 m}{\partial Z^2} + \mu S \frac{\partial m}{\partial S} + p \frac{\partial m}{\partial Z} - rm \\ = (\mu - r)S\Delta. \end{aligned}$$

Once again, we emphasize that all the drift coefficients are from the physical dynamic of the spot process not from risk-adjusted dynamic. For simplicity, we will write

$$\mathcal{L} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \rho\sigma Sq \frac{\partial^2}{\partial S \partial Z} + \frac{1}{2}q^2 \frac{\partial^2}{\partial Z^2} + \mu S \frac{\partial}{\partial S} + p \frac{\partial}{\partial Z}$$

and the equation for the mean becomes

$$\frac{\partial m}{\partial t} + \mathcal{L}m - rm = (\mu - r)S\Delta. \quad (2)$$

We still have to decide on  $\Delta$ . We will choose it to minimize the variance locally, so we can't choose it until we've analyzed the variance in



the next section. Note also that the final condition for (2) will be the payoff for our original option that we are trying to replicate.

This equation for  $m$  was easy to derive, the equation for the variance is a bit harder.

## 5 Analysis of the Variance

The variance  $v(S(t), Z(t), t)$  is defined by

$$v(S(t), Z(t), t) = E_t \left[ (\Pi(t, T) - m(S(t), Z(t), t))^2 \right].$$

We may write

$$\Pi(t, T) - m(S(t), Z(t), t) = (1 - rdt)A_1 + A_2 + O(dt)$$

where

$$\begin{aligned} A_1 &= \Pi(t + dt, T) - (m + dm), \\ A_2 &= dm - \Delta dS. \end{aligned}$$

Also note that  $A_1$  and  $A_2$  are uncorrelated. Therefore

$$v = E_t[(1 - rdt)^2(v + dv) + (dm - \Delta dS)^2] + o(dt)$$

which further reduces to

$$o(dt) = E_t[dv] - 2rvdt + E_t \left[ \left( -\sigma S \Delta dX_1 + \frac{\partial m}{\partial Z} q dX_2 + \frac{\partial m}{\partial S} \sigma S dX_1 \right)^2 \right].$$

The end result is, for an arbitrary  $\Delta$ ,

$$\begin{aligned} 0 &= \frac{\partial v}{\partial t} + \mathcal{L}v - 2rv + \sigma^2 S^2 \left( \frac{\partial m}{\partial S} \right)^2 + 2\rho\sigma Sq \frac{\partial m}{\partial S} \frac{\partial m}{\partial Z} + q^2 \left( \frac{\partial m}{\partial Z} \right)^2 \\ &\quad + \sigma^2 S^2 \Delta^2 - 2\Delta \left( \sigma^2 S^2 \frac{\partial m}{\partial S} + \rho\sigma Sq \frac{\partial m}{\partial Z} \right). \end{aligned} \quad (3)$$

## 6 Choosing $\Delta$ to Minimize the Variance

Only the last two terms in (3) contain  $\Delta$ . We therefore choose  $\Delta$  to minimize this quantity, to ensure that the variance in our portfolio is as small as possible. This gives

$$\Delta = \frac{\partial m}{\partial S} + \frac{\rho q}{\sigma S} \frac{\partial m}{\partial Z}. \quad (4)$$

## 7 The Mean and Variance Equations

Define a risk-adjusted differential operator

$$\mathcal{L}^* = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \rho\sigma Sq \frac{\partial^2}{\partial S \partial Z} + \frac{1}{2} q^2 \frac{\partial^2}{\partial Z^2} + rS \frac{\partial}{\partial S} + p \frac{\partial}{\partial Z}.$$

Substituting (4) into (2) and (3) we get

$$\frac{\partial m}{\partial t} + \mathcal{L}^* m - rm = \frac{\mu - r}{\sigma} \rho q \frac{\partial m}{\partial Z} \quad (5)$$

and

$$\frac{\partial v}{\partial t} + \mathcal{L}v - 2rv + q^2(1 - \rho^2) \left( \frac{\partial m}{\partial Z} \right)^2 = 0. \quad (6)$$

The final conditions for these are obviously the payoff, for  $m(S, Z, T)$ , and zero for  $v(S, Z, T)$ . If the portfolio contains options with different maturities, the equations must satisfy the corresponding jump conditions as well.

Since the final condition for  $v$  is zero and the only ‘forcing term’ in (6) is  $\left( \frac{\partial m}{\partial Z} \right)^2$ , equation (6) shows that the only way we can have a perfect hedge is for either  $q$  to be zero, i.e. deterministic volatility, or to have  $\rho = \pm 1$ . In the latter case the asset and volatility (changes) are perfectly correlated. The solution of (5) is then different from the Black–Scholes solution.

Equation (5) is very much like the pricing equation for stochastic volatility in a risk-neutral setting. It’s rather like having a market price of volatility risk of  $(\mu - r)\rho/\sigma$ . But, of course, the reasoning and model are completely different in our case.

The system of equations is nonlinear (actually two linear equations, coupled by a nonlinear forcing term). We are going to exploit this fact shortly.

## 8 How to Interpret and Use the Mean and Variance

Take an option position in a world with stochastic volatility, and delta hedge as proposed above. Because we cannot eliminate all the risk we cannot be certain how accurate our hedging will be. Think of the final value of the portfolio together with accumulated hedging as being the ‘outcome.’ The distribution of the outcome will generally not be Normal. The shape will depend very much on the option position we are hedging. But we have calculated both the mean and the variance of the hedged portfolio.

If the distribution of profit/loss were Normal then we could interpret the mean and the variance as in Figure 1.

Since this is likely to be one of very many trades, the Central Limit Theorem tells us that only the mean and the variance matter as far as our long-term profitability is concerned.

It is therefore natural to price the contract so as to ensure that it has a specified probability of being profitable. If we made the assumption that the distribution was not too far from Normal then the mean and the variance are sufficient to describe the probabilities of any outcome. If we wanted to be 95% certain that we would make money then we would have to sell the option for

$$m + 1.644853v^{1/2}$$

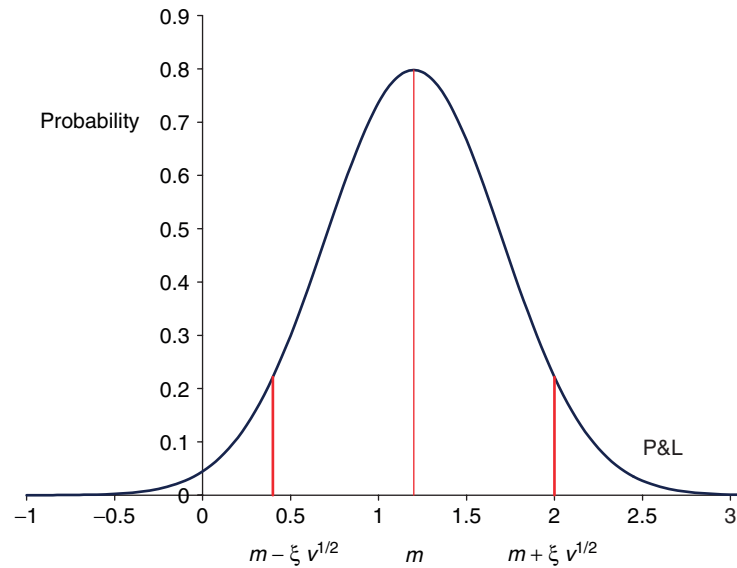


Figure 1: Distribution of profit/loss

or buy it for

$$m - 1.644853v^{1/2}.$$

The 1.644853 comes from the position of the 95th percentile assuming a Normal distribution.

More generally we would price at

$$m \pm \xi v^{1/2},$$

where the  $\xi$  is a personal choice.

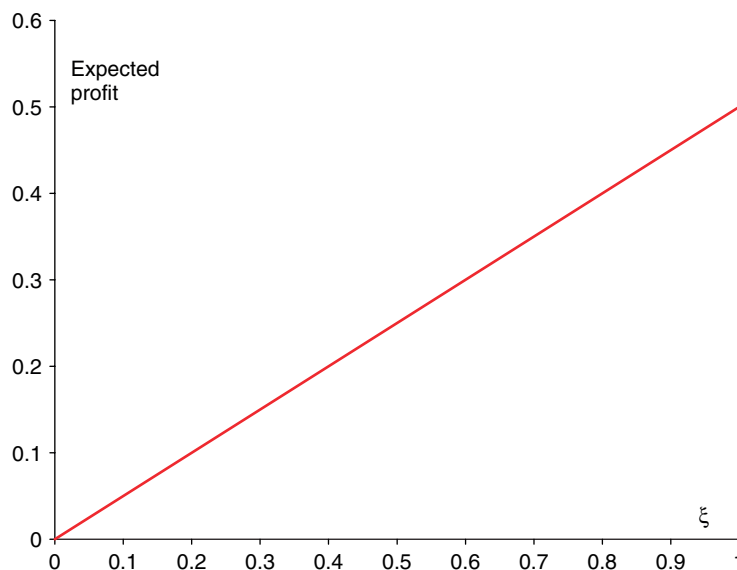


Figure 2: Expected profit from a single trade versus  $\xi$

Clearly the larger  $\xi$  the greater the potential for profit from a single trade, see Figure 2.

However, the larger  $\xi$  the fewer trades, see Figure 3.

The net result is that the total profit potential, being a product of the number of trades and the profit from each trade, is of the form shown in Figure 4. Don't be too greedy or too generous.

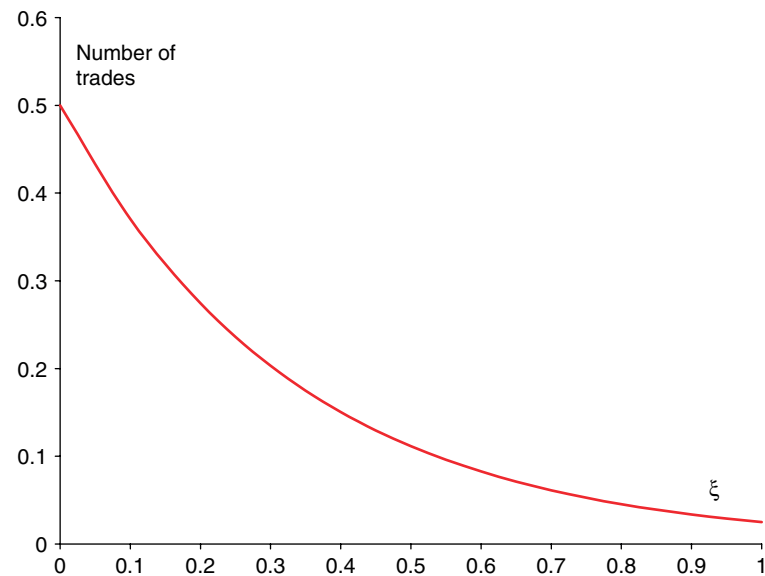


Figure 3: Number of trades versus  $\xi$

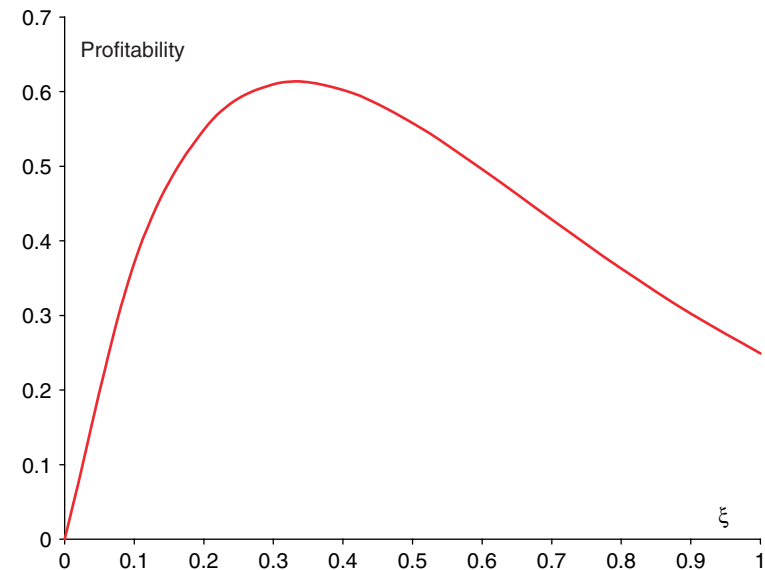


Figure 4: Total profit potential versus  $\xi$

We'll use this idea in the example below, but we will insist that we are within one standard deviation of the mean so that  $\xi = 1$ . This is simply so that we have fewer parameters to carry around.

## 9 Static Hedging and Portfolio Optimization

If we use as our option (portfolio) 'price' the following

$$\text{mean} - (\text{variance})^{1/2} = m - v^{1/2}$$

then we have a non linear model.

Whenever we have a non linear model we have the potential for improving the price by static hedging (see Avellaneda and Parás, 1995, and Wilmott, 2000). This static hedging is, unlike the static hedging of linear problems, completely internally consistent. We will see how this works in the example.

## 10 Example: Valuing and Hedging an Up-and-out Call

In this section, we consider the pricing and hedging of a short up-and-out call. Furthermore, we will consider a special case when the stochastic volatility is parameterized in a classical way:  $\sigma(S, Z, t) = Z$ . Throughout this section, our choice of mean-variance combination is:

$$m - v^{1/2}. \tag{7}$$

First consider a single up-and-out call with barrier located at  $S_u$ . In this case, we solve the equations (5) and (6) subject to:

- (a)  $m(S_u, \sigma, t) = v(S_u, \sigma, t) = 0$  for each  $(\sigma, t) \in (0, \infty) \times [0, T]$  where  $T$  is maturity;
- (b)  $m(S, \sigma, T) = -\max(S - E, 0)$  for each  $(S, \sigma) \in (0, X) \times (0, \infty)$  where  $E$  is the strike;
- (c)  $v(S, \sigma, T) = 0$  for each  $(S, \sigma)$ .

The discontinuity of the payoff at the knock-out barrier makes this position particularly difficult to hedge. In fact this can be easily seen from our equations. Figure 5 and Figure 6 are the pictures of calculated mean and variance respectively with strike at 100, barrier at 110, and expiry in 30 days. We have chosen the model

$$p(\sigma) = 0.8(\sigma^{-1} - 0.2), \quad q(\sigma) = 0.5 \quad \text{with} \quad \rho = 0.$$

Near the barrier,  $\left(\frac{\partial m}{\partial \sigma}\right)^2$  is huge (see Figure 5) and this feeds the variance, being the source term in (6). If the spot  $S$  is 100, and the current spot volatility  $\sigma$  is 20% per annum, the mean is  $-1.1101$  and the variance is  $0.3290$ . Thus if there is no other instrument available in the market, one would price this option at  $\$1.6836$  to match with Equation (7).

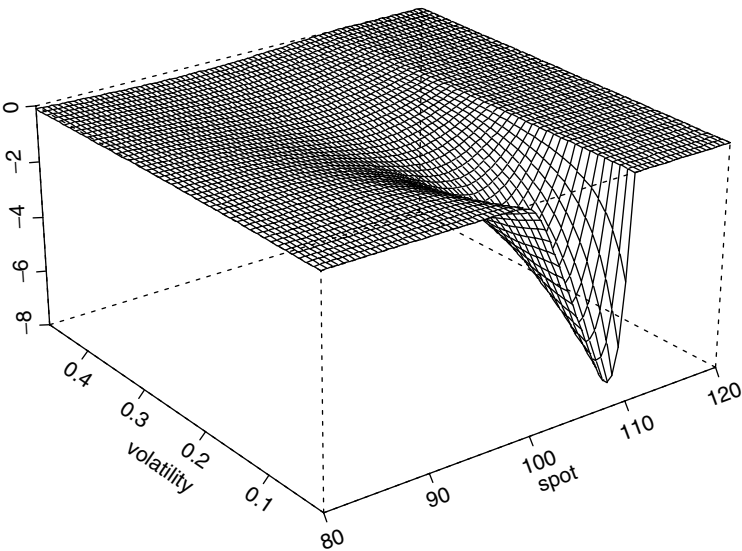


Figure 5: Mean for a single up-and-out call

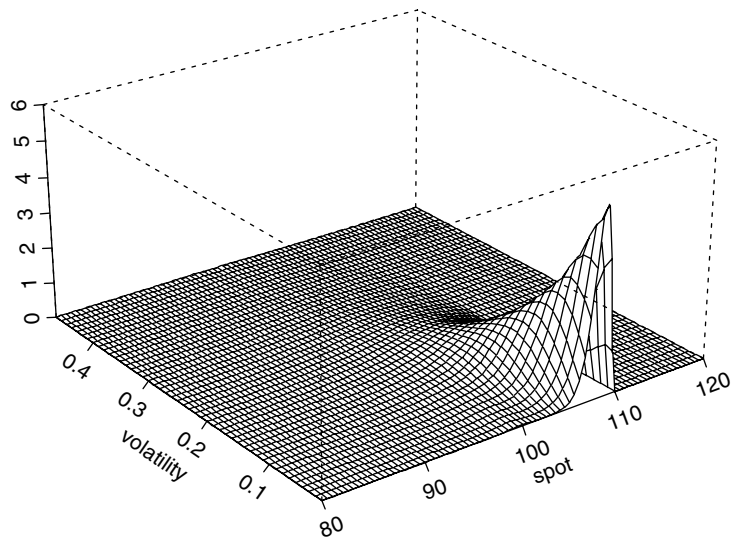


Figure 6: Variance for a single up-and-out call

These results are shown in the table.

	Mean ( $m$ )	Var. ( $v$ )	Value
Unhedged	-1.1101	0.329	1.6836

### 10.1 Static Hedging

Suppose that there are six 30-day vanilla call options available in the market with the following specifications:



Option	1	2	3	4	5	6
Strike	96.62	100.00	104.17	108.70	112.36	116.96
Bid Price	4.6186	2.6774	1.1895	0.4302	0.1770	0.0557
Ask Price	4.6650	2.7043	1.2014	0.4345	0.1788	0.0562

(Aside: These hypothetical market prices were generated by computing the mean of each call option, with

$$d\sigma = \left(\frac{1}{\sigma} - 0.2\right)dt + 0.5 dX_2 \quad (8)$$

where  $X$  is a Brownian motion with respect to the risk-neutral measure. Then 0.5% bid-ask spread was added.)

Now we employ the optimal static vega hedge. Suppose we trade  $(q_1, \dots, q_6)$  of the above instruments and let  $E_i$  be the strikes among the payoffs. Furthermore, let  $(m^{(0)}, v^{(0)})$  be the mean variance pair after knock out and  $(m^{(1)}, v^{(1)})$  be that before knock out. Then  $(m^{(i)}, v^{(i)})$ ,  $i = 0, 1$ , satisfy the equations (5) and (6) subject to:

- (a)  $m^{(1)}(110, \sigma, t) = m^{(0)}(110, \sigma, t)$  and  $v^{(1)}(110, \sigma, t) = v^{(0)}(110, \sigma, t)$  for each  $(\sigma, t)$  in  $(0, \infty) \times [0, T)$ ;
- (b)  $m^{(0)}(S, \sigma, t) = \sum_{i=1}^6 q_i \max(S - E_i, 0)$  for each  $(S, \sigma) \in (0, \infty) \times (0, \infty)$ ;
- (c)  $m^{(1)}(S, \sigma, T) = \sum_{i=1}^6 q_i \max(S - E_i, 0) - \max(S - 100, 0)$  for each  $(S, \sigma)$  in  $(0, 110) \times (0, \infty)$ ;
- (d)  $v^{(1)}(S, \sigma, T) = v^{(0)}(S, \sigma, T) = 0$  for each  $(S, \sigma)$  in  $(0, \infty) \times (0, \infty)$ .

Thus  $m^{(1)}(S, \sigma, 0)$  stands for the mean of the cashflows excluding the up-front premium. We find a  $(q_1, \dots, q_6)$  that maximizes:

$$m^{(1)}(S, \sigma, 0) - \sqrt{v^{(1)}(S, \sigma, 0)} - \sum_{i=1}^6 p(q_i)$$

where  $p(q_i)$  is the market price of trading  $q_i$  shares of strike  $E_i$ . In the case of  $S = 100$  and  $\sigma = 0.2$ , our optimal choice for vega hedge is given by:

Option	1	2	3	4	5	6
Strike	96.62	100.00	104.17	108.70	112.36	116.96
Quantity	0.0000	-1.1688	1.0207	3.1674	-3.6186	0.8035

The cost of this hedge position is \$1.1863. Figure 7 and Figure 8 are the pictures of  $m^{(1)}$  and  $v^{(1)}$  after the optimal static vega hedge. After the optimal static vega hedge, the mean is 0.0398 and the variance is reduced to 0.0522. Thus the price for the up-and-out call that matches with our mean-variance combination (7) is \$1.3752 ( $1.1863 - 0.0398 + \sqrt{0.0522}$ ). In the risk-neutral set-up (8), the price for this up-and-out call is \$1.1256. The difference is mainly comes from the standard deviation term (variance<sup>1/2</sup>) in (7) which is  $\sqrt{0.0522} = 0.2286$ .

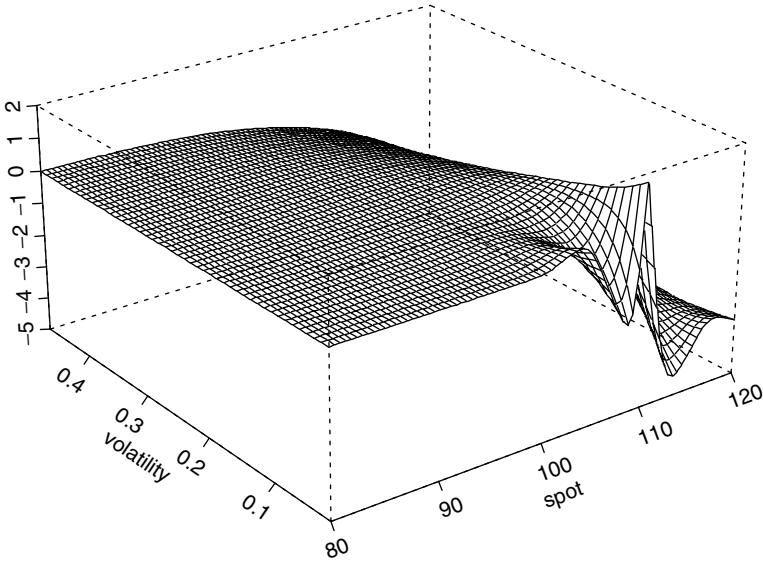


Figure 7: Mean of portfolio after optimal static vega hedging

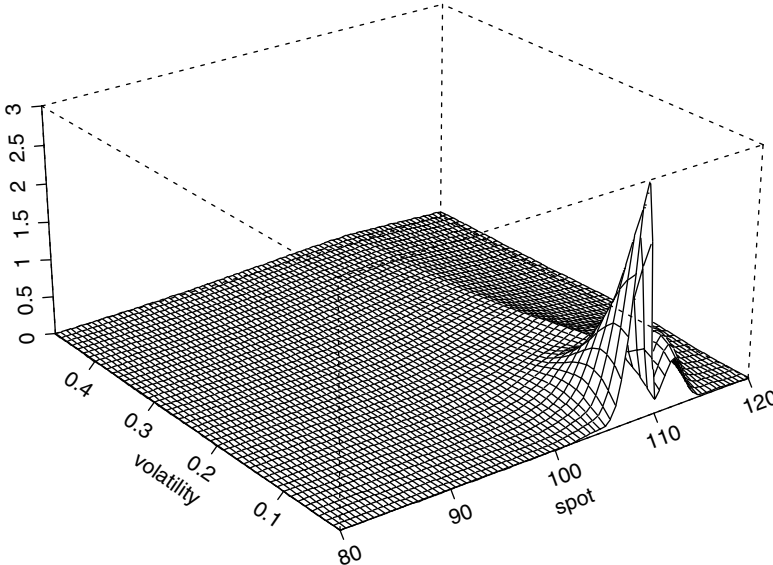


Figure 8: Variance of portfolio after optimal static vega hedging

These results are shown in the table.

	Mean ( $m$ )	Var. ( $v$ )	Hedge	Value
Unhedged	-1.1101	0.329		1.6836
Hedged	0.0398	0.0522	1.1863	1.3752

By statically hedging we have reduced the price at which we can safely sell the option, from 1.6836 to 1.3752, while still making money 84% of the time. Alternatively, we can still sell the option for 1.6836 and make even more profit.

At the same time the variance has been dramatically reduced so that we are less exposed to volatility risk than if we had not statically hedged the position.

## 11 Other Definitions of 'Value'

In the above example we have statically hedged so as to find the best value according to our definition of value. This is by no means the only static hedging strategy. One can readily imagine different players having different criteria.

Obvious strategies that spring to mind are as follows.

- Minimize variance, that is minimize the function  $v$ . This has the effect of reducing model risk as much as possible using all available instruments (the underlying and all traded options). This may be a strategy adopted by the sell side.
- Maximize the return-risk ratio. This is perhaps more of a buy-side strategy, for maximizing Sharpe ratio, for example.

## 12 Summary

Constructing a risk-neutral model to fit the market prices of exchange traded options consistently over a reasonable time period is a difficult task. Putting aside the fundamental question of whether the axiomatic risk-neutral model for stochastic volatility is legitimate or not, we must face potential financial losses due to re-calibration. In this paper we have taken another approach. We first evaluate the mean and variance of the discounted future cashflow and then find market instruments that reduce the volatility risk optimally.

We've set this problem up in a mean-variance framework but it could easily be extended to a more general utility theory approach.

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