Computation in an Asymptotic Expansion Method *

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Abstract

An asymptotic expansion scheme in finance initiated by Kunitomo and Takahashi [15] and Yoshida[68] is a widely applicable methodology for analytic approximation of the expectation of a certain functional of diffusion processes. [46], [47] and [53] provide explicit formulas of conditional expectations necessary for the asymptotic expansion up to the third order. In general, the crucial step in practical applications of the expansion is calculation of conditional expectations for a certain kind of Wiener functionals. This paper presents two methods for computing the conditional expectations that are powerful especially for high order expansions: The first one, an extension of the method introduced by the preceding papers presents a general scheme for computation of the conditional expectations and show the formulas useful for expansions up to the fourth order explicitly. The second one develops a new calculation algorithm for computing the coefficients of the expansion through solving a system of ordinary differential equations that is equivalent to computing the conditional expectations. To demonstrate their effectiveness, the paper gives numerical examples of the approximation for λ-SABR model up to the fifth order and a cross-currency Libor market model with a general stochastic volatility model of the spot foreign exchange rate up to the fourth order.

1 Introduction

This paper presents two alternative schemes for computation in an asymptotic expansion approach based on Watanabe theory(Watanabe [66]) in Malliavin

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calculus by extending the preceding papers and also by developing a new calculation algorithm.

To our best knowledge, the asymptotic expansion is first applied to finance for evaluation of an average option that is a popular derivative in commodity markets. [15] and [46] derive the approximation formulas for an average option by an asymptotic method based on log-normal approximations of an average price distribution when the underlying asset price follows a geometric Brownian motion. [68] applies a formula derived more generally by the asymptotic expansion of small diffusion processes. Thereafter, the asymptotic expansion have been applied to a broad class of problems in finance: See [47], [48], [49], [50], Kunitomo and Takahashi [16], [17], [18], [19], Kawai [11], Matsuoka, Takahshi and Uchida [31], Takahashi and Matsushima [51], Takahashi and Saito [52], Takahashi and Yoshida [57], [58], Kobayashi, Takahashi and Tokioka [13], Muroi [33], Osajima [38], Takahashi and Uchida [56], Kunitomo and Kim [14], Kawai and Jäckel [12], and [53], [54], [55].

For other asymptotic methods in finance which do not depend on Watanabe theory, see also Fouque, Papanicolaou and Sircar [5], [6], Henry-Labordere [24], [25], [26], Kusuoka and Osajima [20], Osajima [39] and Siopacha and Teichmann [45].

In the application of the asymptotic expansion based on Watanabe theory, they calculated certain conditional expectations which appear in their expansions and play a key role in computation, by the formulas up to the third order given explicitly in [46], [47] and [53]. In many applications, these formulas give sufficiently accurate approximation, but in some cases, for example in the cases with long maturities or/and with highly volatile underlying variables, the approximation up to the third order may not provide satisfactory accuracies. Thus, the formulas for the higher order computation are desirable. But to our knowledge, asymptotic expansion formulas higher than the third order have not been given yet. This paper provides the general procedures for the explicit computation of conditional expectations in the asymptotic expansion and show the formulas for the approximation up to the fourth order. Moreover, we develop another calculation algorithm which enables us to derive high order approximation formulas in an automatic manner. As a consequence, our approximation generally shows sufficient accuracy with computation of high order expansions, which is confirmed by numerical experiments.

In the following sections, after a brief explanation of the asymptotic expansion in Section 2, Section 3 will provide a computation procedure explicitly for conditional expectations appearing in the expansion and show the formulas for expansions up to the fourth order. Moreover, Section 4 will introduce our new alternative computation algorithm for the asymptotic expansion and derive the fourth order asymptotic expansion formula. Finally, Section 5 will apply our algorithms described in the previous sections to the concrete financial models, and confirm the effectiveness of the higher order expansions by numerical examples in λ-SABR model and a cross-currency Libor market model with a general stochastic volatility model of the spot foreign exchange rate.

2 An Asymptotic Expansion in a General Markovian Setting

This section briefly describes an asymptotic expansion method in a general Markovian setting.

Let (W, P) be the r-dimensional Wiener space. We consider a d-dimensional diffusion process $X_t^{(\epsilon)} = (X_t^{(\epsilon),1}, \cdots, X_t^{(\epsilon),d})$ which is the solution to the following stochastic differential equation:

$$
\begin{aligned}
dX_t^{(\epsilon),j} &= V_0^j(X_t^{(\epsilon)},\epsilon)dt + \epsilon V^j(X_t^{(\epsilon)})dW_t \quad (j=1,\cdots,d) \\
X_0^{(\epsilon)} &= x_0 \in \mathbf{R}^d
\end{aligned} \tag{1}
$$

where $W = (W^1, \cdots, W^r)$ is a r-dimensional standard Wiener process, and $\epsilon \in (0,1]$ is a known parameter. Let

Suppose that $V_0 = (V_0^1, \cdots, V_0^d) : \mathbf{R}^d \times (0,1] \mapsto \mathbf{R}^d$ and $V = (V^1, \cdots, V^d)$: $\mathbf{R}^d \mapsto \mathbf{R}^d \otimes \mathbf{R}^r$ satisfy some regularity conditions.(e.g. V_0 and V are smooth functions with bounded derivatives of all orders.)

Next, suppose that a function $g : \mathbf{R}^d \mapsto \mathbf{R}$ to be smooth and all derivatives have polynomial growth orders. Then, a smooth Wiener functional $g(X_T^{(\epsilon)})$ has its asymptotic expansion;

$$
g(X_T^{(\epsilon)}) \sim g_{0T} + \epsilon g_{1T} + \cdots
$$

in L^p for every $p > 1$(or in $\mathbf{D}^\infty$) as $\epsilon \downarrow 0$. The coefficients in the expansion $g_{nT} \in \mathbf{D}^\infty(n = 0, 1, \cdots)$ can be obtained by Taylor's formula and represented based on multiple Wiener-Itô integrals. Here, $\mathbf{D}^\infty$ denotes the set of smooth Wiener functionals. See chapter V of Ikeda and Watanabe[8] for the detail.

Let $A_{kt} = \frac{1}{k!}\frac{\partial^k X_t^{(\epsilon)}}{\partial \epsilon^k}|_{\epsilon=0}$ and A_{kt}^j, $j = 1, \cdots, d$ denote the j-th elements of A_{kt}. In particular, A_{1t} is represented by

$$
A_{1t} = \int_0^t Y_t Y_u^{-1}\left(\partial_\epsilon V_0(X_u^{(0)}, 0)du + V(X_u^{(0)})dW_u\right) \tag{2}
$$

where Y denotes the solution to the differential equation;

$$
dY_t = \partial V_0(X_t^{(0)}, 0)Y_t dt;\ Y_0 = I_d.
$$

Here, ∂V_0 denotes the $d \times d$ matrix whose (j,k)-element is $\partial_k V_0^j = \frac{\partial V_0^j(x,\epsilon)}{\partial x_k}$, V_0^j is the j-th element of V_0, and I_d denotes the $d \times d$ identity matrix.

For $k \geq 2$, A_{kt}^j, $j = 1, \cdots, d$ is recursively determined by the following:

$$
\begin{aligned}
A_{kt}^j &= \frac{1}{k!}\int_0^t \partial_\epsilon^k V_0^j(X^{(0)}, 0)du \\
&\quad + \sum_{l=1}^{k} \sum_{\vec{l}_\beta, \vec{d}_\beta}^{(l)} \frac{1}{(k-l)!}\frac{1}{\beta!}\int_0^t \left(\prod_{j=1}^{\beta} A_{l_j u}^{d_j}\right) \partial_{\vec{d}_\beta}^\beta \partial_\epsilon^{k-l} V_0^j(X_u^{(0)}, 0)du \\
&\quad + \sum_{\vec{l}_\beta, \vec{d}_\beta}^{(k-1)} \frac{1}{\beta!}\int_0^t \left(\prod_{j=1}^{\beta} A_{l_j u}^{d_j}\right) \partial_{\vec{d}_\beta}^\beta V^j(X_u^{(0)})dW_u
\end{aligned} \tag{3}
$$

where $\partial_\epsilon^l = \frac{\partial^l}{\partial \epsilon^l}$, $\partial_{\vec{d}_\beta}^\beta = \frac{\partial^\beta}{\partial x_{d_1} \cdots \partial x_{d_\beta}}$,

$$
L_{n,\beta} = \left\{ \vec{l}_\beta = (l_1, \cdots, l_\beta); \sum_{j=1}^{\beta} l_j = n, l_j \geq 1, j = 1, \cdots, \beta \right\} \tag{4}
$$

and

$$
\sum_{\vec{l}_\beta, \vec{d}_\beta}^{(n)} = \sum_{\beta=1}^{n} \sum_{\vec{l}_\beta \in L_{n,\beta}} \sum_{\vec{d}_\beta \in \{1,\cdots,d\}^\beta}
$$

for $n \geq 1$, and

$$\sum_{\vec{l}_\beta, \vec{d}_\beta}^{(0)} = \sum_{\beta=0} \sum_{\vec{l}_0=(\emptyset)} \sum_{\vec{d}_0=(\emptyset)} .$$

Then, g_{0T} and g_{1T} can be written as

$$\begin{aligned} g_{0T} &= g(X_T^{(0)}), \\ g_{1T} &= \sum_{j=1}^{d} \partial_j g(X_T^{(0)}) A_{1T}^j. \end{aligned}$$

For $n \geq 2$, g_{nT} is expressed as follows:

$$g_{nT} = \sum_{\vec{l}_\beta \vec{d}_\beta}^{(n)} \frac{1}{\beta!} \partial_{\vec{d}_\beta}^{\beta} g(X_T^{(0)}) A_{l_1 T}^{d_1} \cdots A_{l_\beta T}^{d_\beta}. \tag{5}$$

Next, normalize $g(X_T^{(\epsilon)})$ to

$$G^{(\epsilon)} = \frac{g(X_T^{(\epsilon)}) - g_{0T}}{\epsilon}$$

for $\epsilon \in (0, 1]$. Then,

$$G^{(\epsilon)} \sim g_{1T} + \epsilon g_{2T} + \cdots$$

in L^p for every $p > 1$(or in $\mathbf{D}^\infty$). Moreover, let

$$\hat{V}(x,t) = (\partial g(x))' [Y_T Y_t^{-1} V(x)]$$

and make the following assumption:

$$\text{(Assumption 1)} \quad \Sigma_T = \int_0^T \hat{V}(X_t^{(0)}, t) \hat{V}(X_t^{(0)}, t)' dt > 0.$$

Note that g_{1T} follows a normal distribution with variance Σ_T; the density function of g_{1T} denoted by $f_{g_{1T}}(x)$ is given by

$$f_{g_{1T}}(x) = \frac{1}{\sqrt{2\pi\Sigma_T}} \exp\left(-\frac{(x-C)^2}{2\Sigma_T} \right)$$

where

$$C := \left(\partial g(X_T^{(0)}) \right)' \int_0^T Y_T Y_t^{-1} \partial_\epsilon V_0(X_t^{(0)}, 0) dt. \tag{6}$$

Hence, Assumption 1 means that the distribution of g_{1T} does not degenerate. In application, it is easy to check this condition in most cases. Hereafter, Let $\mathcal{S}$ be the real Schwartz space of rapidly decreasing $\mathbf{C}^\infty$-functions on $\mathbf{R}$ and $\mathcal{S}'$ be its dual space that is the space of the Schwartz tempered distributions. Next, take $\Phi \in \mathcal{S}'$. Then, by Watanabe theory(Watanabe [66], Yoshida [67]) a generalized Wiener functional $\Phi(G^{(\epsilon)})$ has an asymptotic expansion in $\mathbf{D}^{-\infty}$ as $\epsilon \downarrow 0$ where $\mathbf{D}^{-\infty}$ denotes the set of generalized Wiener functionals. See chapter V of Ikeda and Watanabe[8] for the detail. Hence, the expectation of $\Phi(G^{(\epsilon)})$ is expanded around $\epsilon = 0$ as follows: For $N = 0, 1, 2, \cdots$,

$$
\begin{aligned}
\mathbf{E}[\Phi(G^{(\epsilon)})] &= \sum_{j=0}^{N} \epsilon^j \sum_{m=0}^{j} \frac{1}{m!} \mathbf{E}\left[\Phi^{(m)}(g_{1T}) \left\{ \sum_{k \in K_{j,m}} C^{j,m,k} \prod_{n=1}^{j-m+1} g_{(n+1)T}^{k_n} \right\}\right] + o(\epsilon^N) \\
&= \sum_{j=0}^{N} \epsilon^j \sum_{m=0}^{j} \frac{1}{m!} \mathbf{E}\left[\Phi^{(m)}(g_{1T}) \sum_{k \in K_{j,m}} C^{j,m,k} \mathbf{E}\left[X^{j,m,k} \middle| g_{1T} \right]\right] + o(\epsilon^N) \\
&= \sum_{j=0}^{N} \epsilon^j \sum_{m=0}^{j} \frac{1}{m!} \int_{\mathbf{R}} \Phi^{(m)}(x) \sum_{k \in K_{j,m}} C^{j,m,k} \mathbf{E}\left[X^{j,m,k} \middle| g_{1T} = x \right] f_{g_{1T}}(x) dx + o(\epsilon^N) \\
&= \sum_{j=0}^{N} \epsilon^j \sum_{m=0}^{j} \frac{1}{m!} \int_{\mathbf{R}} \Phi(x) \sum_{k \in K_{j,m}} C^{j,m,k} (-1)^m \frac{\partial^m}{\partial x^m} \{\mathbf{E}\left[X^{j,m,k} \middle| g_{1T} = x \right] f_{g_{1T}}(x)\} dx + o(\epsilon^N)
\end{aligned} \tag{7}
$$

where $\Phi^{(m)}(g_{1T}) = \frac{\partial^m \Phi(x)}{\partial x^m}\Big|_{x=g_{1T}}$,

$$
K_{j,m} = \left\{ (k_1, \cdots, k_{j-m+1}); k_n \geq 0, \sum_{n=1}^{j-m+1} k_n = m, \sum_{n=1}^{j-m+1} n k_n = j \right\},
$$

$$
\begin{aligned}
X^{j,m,k} &= \prod_{n=1}^{j-m+1} g_{(n+1)T}^{k_n}, \\
C^{j,m,k} &= \prod_{n=1}^{j-m+1} \frac{m!}{k_1! \cdots k_{j-m+1}!}.
\end{aligned}
$$

3 Computation of Conditional Expectations

3.1 Procedures of Computations

In the previous section, we have

$$
\mathbf{E}[\Phi(G^{(\epsilon)})] = \sum_{j=0}^{N} \epsilon^j \sum_{m=0}^{j} \frac{1}{m!} \int_{\mathbf{R}} \Phi(x) \sum_{k \in K_{j,m}} C^{j,m,k} (-1)^m \frac{\partial^m}{\partial x^m} \{\mathbf{E}\left[X^{j,m,k} \middle| g_{1T} = x \right] f_{g_{1T}}(x)\} dx + o(\epsilon^N). \tag{8}
$$

Then, if we obtain conditional expectations appearing in this expression explicitly, it can be easily calculated since g_{1T} follows a normal distribution. In particular, letting Φ be δ_x, the delta function at $x \in \mathbf{R}$, the asymptotic expansion of the density function of $G^{(\epsilon)}$ can be obtained as in (28) in the next section.

Here we describe the procedures of evaluating these conditional expectations.

At the beginning of this subsection, we state the following proposition playing an important role in the evaluation.

Proposition 1 *Let $J_n(f_n)$ denote the n-times iterated Itô integral of $L^2(\mathbf{T}^n)$-function f_n:*

$$
J_n(f_n) := \int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} f_n(t_1, \cdots, t_n) dW_{t_n} \cdots dW_{t_2} dW_{t_1}
$$

for $n \geq 1$ and $J_0(f_0) := f_0(constant)$.

Then, its expectation conditional on $J_1(q) = x$ is given by

$$\mathbf{E}[J_n(f_n)|J_1(q) = x] = \left(\int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} f_n(t_1, \cdots, t_n) q(t_1) \cdots q(t_n) dt_n \cdots dt_2 dt_1\right) \frac{H_n(x; \|q\|^2_{L^2(\mathbf{T})})}{(\|q\|^2_{L^2(\mathbf{T})})^n} \tag{9}$$

where $\mathbf{T} = [0, T]$, $t_i \in \mathbf{T}(i = 1, 2, \cdots, n)$ and $H_n(x; \Sigma)$ is the Hermite polynomial of degree n which is defined as

$$H_n(x; \Sigma) := (-\Sigma)^n e^{x^2/2\Sigma} \frac{d^n}{dx^n} e^{-x^2/2\Sigma}.$$

(proof) See Section 3.2.□

Next, we show how to compute the conditional expectations in (8). In the rest of this subsection, we assume $\partial_\epsilon V_0(X_t^{(0)}, 0) \equiv (0, \cdots, 0)$ with no loss of generality, and set $q(t) = a_t = (\partial g(X_T^{(0)}))' [Y_T Y_t^{-1} V(X_t^{(0)})]$ (then $J_1(q) = g_{1T}$ and $\|q\|^2_{L^2(\mathbf{T})} = \Sigma_T$). If this assumption is not satisfied, we can obtain almost the same result by taking conditional expectations with respect to

$$\hat{g}_{1T} := g_{1T} - C$$

instead of g_{1T}, where

$$C := \left(\partial g(X_T^{(0)})\right)' \int_0^T Y_T Y_t^{-1} \partial_\epsilon V_0(X_t^{(0)}, 0) dt.$$

The procedures consist of three steps.

1. The way to derive an expansion of $A_{lt}^i = \frac{\partial^l X_t^{(\epsilon)}}{\partial \epsilon^l}|_{\epsilon=0}$ is explained. (The definition of A_{lt}^i in this section is slightly different from that in the previous section. ($A_{lt}^i = \frac{1}{k!} \frac{\partial^l X_t^{(\epsilon)}}{\partial \epsilon^l}|_{\epsilon=0}$ in p.3.)) In this stage, there are two alternative ways.

   - In one way, as in Lemma 2 in Section 3.2, A_{lT}^i can be expanded as a summation of at most l iterated Itô integrals whose integrands are a family of symmetric $L^2(\mathbf{T}^l)$-functions $\{\hat{f}_{l'}^{i,l}\}_{l'=1}^l$:

   $$A_{lt}^i = \sum_{l'=1}^l J_{l'}(\hat{f}_{l'}^{i,l}) \tag{10}$$

   The integrand of l'-times iterated Itô integral in this expansion is given by the expectation of the l'-th Malliavin derivative of A_{lT}^i:

   $$\hat{f}_{l'}^{i,l}(t_1, \cdots, t_{l'}) = \mathbf{E}\left[D_{t_1, \cdots, t_{l'}} A_{lT}^i\right]. \tag{11}$$

   - In fact, as we can see in (3) every A_{lT}^i is given by finite operations of multiplication, (Lebesgue) integration with respect to time parameters and stochastic integrations. Then, the alternative expansion(up to the l-th order) of A_{lT}^i can be directly calculated via iterated use of Itô's formula:

   $$A_{lT}^i = \sum_{l'=0}^l J_{l'}(f_{l'}^{i,l}) \tag{12}$$

- We here briefly advert to the relationship between $\hat{f}^{i,l}_{l'}$ and $f^{i,l}_{l'}$. Note that A^i_{lT} has its Wiener-Chaos expansion as in the proof of Lemma 2 which is described in Section 3.2

$$A^i_{lt} = \sum_{l'=1}^{l} I_{l'}(\tilde{f}^{i,l}_{l'}) \tag{13}$$

and that the integrand of l'-th order multiple Wiener-Itô integral is given by

$$\tilde{f}^{i,l}_{l'}(t_1,\cdots,t_{l'}) = \frac{1}{l'!}\mathbf{E}\left[D_{t_1,\cdots,t_{l'}}A^i_{lT}\right].$$

Then, due to the relationship between an iterated Itô integral and a multiple Wiener-Itô integral of the same order shown in Lemma 1 in Section 3.2, $\hat{f}^{i,l}_{l'} = l'!\tilde{f}^{i,l}_{l'}$ actually coincides with a symmetrization(unnormalized with respect to its norm) of $f^{i,l}_{l'}$;

$$\hat{f}^{i,l}_{l'}(t_1,\cdots,t_{l'}) = \sum_{\sigma_{l'}} 1_{\{t_{\sigma_{l'}(1)}\geq\cdots\geq t_{\sigma_{l'}(l')}\}} f^{i,l}_{l'}(t_{\sigma_{l'}(1)},\cdots,t_{\sigma_{l'}(l')})$$

where the summation is taken over all permutations $\sigma_{l'}$: $\{1,2,\cdots,l'\}\mapsto\{1,2,\cdots,l'\}$.

2. From the expansion of A^i_{lT}, we derive that of g_{nT}. Recall that for $n\geq 1$

$$g_{nT} = \sum_{\vec{s}\in S_n}\left(\frac{n!}{s_1!\cdots s_n!}\right)\prod_{l=1}^{n}\left(\frac{1}{l!}\right)^{s_l}\sum_{\vec{p}^{s_l}\in P_{s_l}}\left(\frac{s_l!}{p^{s_l}_1!\cdots p^{s_l}_d!}\right)\partial_1^{p^{s_l}_1}\cdots\partial_d^{p^{s_l}_d}g(X^{(0)}_T)\prod_{i=1}^{d}\left(A^i_{lT}\right)^{p^{s_l}_i}$$

where

$$S_n = \left\{\vec{s}=(s_1,\cdots,s_n);\ s_l\geq 0,\ \sum_{l=1}^{n} l s_l = n\right\} \text{ and } P_s = \left\{\vec{p}^s=(p^s_1,\cdots,p^s_d);\ p^s_i\geq 0,\ \sum_{i=1}^{d}p^s_i = s\right\}.$$

Then, the expansions of g_{nT} are obtained by applying Itô's formula iteratively. Moreover, noting that the highest order of the expansion of g_{nT} is n, g_{nT} is expressed as

$$g_{nT} = \sum_{i=0}^{n} J_i(f^{g_n}_i)$$

with integrands $\{f^{g_n}_i\}_{i=1}^n$ obtained via Itô's formula.

3. Again by iterative applications of Itô's formula to

$$X^{j,m,k} = \prod_{n=1}^{j-m+1} g^{k_n}_{(n+1)T}$$

where $k = (k_1,\cdots,k_{j-m+1})\in K_{j,m} = \left\{(k_1,\cdots,k_{j-m+1}); k_n\geq 0,\ \sum_{n=1}^{j-m+1}k_n = m,\ \sum_{n=1}^{j-m+1}nk_n = j\right\}$,

we now have the expansion of $X^{j,m,k}$ in (8) with a finite number of terms as

$$X^{j,m,k} = \sum_{n'=0}^{j+m} J_{n'}(f^{j,m,k}_{n'}) \tag{14}$$

with integrands $\{f_{n'}^{j,m,k}\}_{n'=1}^{j+m}$ which can be deduced from $\{\hat{f}_{l'}^{i,l}\}_{l'=1}^{l}$ in (10) or $\{f_{l'}^{i,l}\}_{l'=1}^{l}$ in (12). Note that this expansion ends with the $(j+m)$-times iterated integral since the highest order of the expansion of $X^{j,m,k}$ can be obtained through the following calculation;

$$\sum_{n=1}^{j-m+1}(n+1)k_n = \sum_{n=1}^{j-m+1} nk_n + \sum_{n=1}^{j-m+1} k_n = j+m.$$

4. From Proposition 1, we conclude that the conditional expectations in (8) are given by

$$\mathbf{E}\left[X^{j,m,k}\middle| g_{1T}=x\right] = \sum_{n'=0}^{j+m}\left(\int_0^T\int_0^{t_1}\cdots\int_0^{t_{n'-1}} f_{n'}^{j,m,k}(t_1,\cdots,t_{n'})q(t_1)\cdots q(t_{n'})dt_{n'}\cdots dt_2dt_1\right)\frac{H_{n'}(x;\Sigma_T)}{\Sigma_T^{n'}}. \tag{15}$$

Example 1

At the end of this subsection we show a simple example evaluating $X^{1,1,(1)} = g_{2T}$ in order to make these procedures clear. Let consider the case when $d = r = 1$ and $V_0(x,\epsilon)\equiv 0$. In this case g_{2T} is given by

$$g_{2T} = \frac{1}{2}\partial^2 g(X_T^{(0)})(A_{1T})^2 + \frac{1}{2}\partial g(X_T^{(0)})A_{2T}$$

where

$$\begin{aligned} A_{1T} &= \int_0^T V(X_u^{(0)})dW_u, \\ A_{2T} &= 2\int_0^T \partial V(X_u^{(0)})A_{1u}dW_u. \end{aligned}$$

Then, it can be decomposed as the sum of $J_n(\cdot)$ by Itô's formula;

$$g_{2T} = J_0(f_0^{g_2}) + J_2(f_2^{g_2})$$

where

$$\begin{aligned} f_0^{g_2} &= \frac{1}{2}\partial^2 g(X_T^{(0)})\int_0^T V(X_u^{(0)})^2du, \\ f_2^{g_2}(t_1,t_2) &= \partial^2 g(X_T^{(0)})V(X_{t_2}^{(0)})V(X_{t_1}^{(0)}) + \partial g(X_T^{(0)})V(X_{t_2}^{(0)})\partial V(X_{t_1}^{(0)}). \end{aligned}$$

From Proposition 1, it follows that

$$\begin{aligned} \mathbf{E}\left[X^{1,1,(1)}|g_{1T}=x\right] &= \Bigg(\partial^2 g(X_T^{(0)})\partial g(X_T^{(0)})^2\int_0^T\int_0^{t_1}V(X_{t_2}^{(0)})^2dt_2V(X_{t_1}^{(0)})^2dt_1 \\ &\quad + \partial g(X_T^{(0)})^3\int_0^T\int_0^{t_1}V(X_{t_2}^{(0)})^2dt_2\partial V(X_{t_1}^{(0)})V(X_{t_1}^{(0)})dt_1\Bigg)\frac{H_2(x;\Sigma_T)}{\Sigma_T^2} \\ &\quad +\frac{1}{2}\partial^2 g(X_T^{(0)})\int_0^T V(X_u^{(0)})^2du. \end{aligned}$$

3.2 Proof of Lemmas and Proposition in Section 3.1

In this subsection we introduce and prove the important proposition and lemmas used in Section 3.1.

Proposition 1 *The expectation of n-times iterated Itô integral $J_n(f_n)$ conditional on $J_1(q) = x$ is given by*

$$\mathbf{E}[J_n(f_n)|J_1(q) = x] = \left(\int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} f_n(t_1, \cdots, t_n) q(t_1) \cdots q(t_n) dt_n \cdots dt_2 dt_1 \right) \frac{H_n(x; \|q\|^2_{L^2(\mathbf{T})})}{(\|q\|^2_{L^2(\mathbf{T})})^n}.$$

(proof) This can be considered as a version of Proposition 3 of Nualart, *Ü*stünel and Zakai [36].

Let $I_n(\hat{f})$ denote the multiple Wiener-Itô integral of n-th order of its integrand $\hat{f} \in L^2_{sym}(\mathbf{T}^n)$, that is $\hat{f}$ is an element of the space of square-integrable symmetric functions from $\mathbf{T}^n$ to $\mathbf{R}$.

Then, from Proposition 3 of [36], we know

$$\mathbf{E}[I_n(\hat{f})|I_1(q) = x] = \left(\int \cdots \int_{\mathbf{T}^n} \hat{f}(t_1, \cdots, t_n) q(t_1) \cdots q(t_n) dt_n \cdots dt_2 dt_1 \right) \frac{H_n(x; \|q\|^2_{L^2(\mathbf{T})})}{(\|q\|^2_{L^2(\mathbf{T})})^n}.$$

Substituting a symmetrization of f_n defined as in (17) in Lemma 1 below, we obtain the result:

$$\begin{aligned}
& \mathbf{E}[J_n(f_n)|J_1(q) = x] \\
= & \mathbf{E}[I_n(\hat{f}_n)|I_1(q) = x] \\
= & \left(\int \cdots \int_{\mathbf{T}^n} \hat{f}_n(t_1, \cdots, t_n) q(t_1) \cdots q(t_n) dt_n \cdots dt_2 dt_1 \right) \frac{H_n(x; \|q\|^2_{L^2(\mathbf{T})})}{(\|q\|^2_{L^2(\mathbf{T})})^n} \\
= & \frac{1}{n!} \sum_{\sigma_n} \left(\int \cdots \int_{\mathbf{T}^n} 1_{\{t_{\sigma_n(1)} \geq \cdots \geq t_{\sigma_n(n)}\}} f_n(t_{\sigma_n(1)}, \cdots, t_{\sigma_n(n)}) q(t_1) \cdots q(t_n) dt_n \cdots dt_2 dt_1 \right) \frac{H_n(x; \|q\|^2_{L^2(\mathbf{T})})}{(\|q\|^2_{L^2(\mathbf{T})})^n} \\
= & \left(\int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} f_n(t_1, \cdots, t_n) q(t_1) \cdots q(t_n) dt_n \cdots dt_2 dt_1 \right) \frac{H_n(x; \|q\|^2_{L^2(\mathbf{T})})}{(\|q\|^2_{L^2(\mathbf{T})})^n}. \square
\end{aligned}$$

The following lemma gives us the relationship between the iterated Itô integral and the multiple Wiener-Itô integral of the same order.

Lemma 1 *For any $L^2(\mathbf{T}^n)$-function f_n which is not necessarily symmetric it holds that*

$$J_n(f_n) = I_n(\hat{f}_n) \tag{16}$$

where $\hat{f}_n$ is a symmetrization of f_n defined by

$$\hat{f}_n(t_1, \cdots, t_n) := \frac{1}{n!} \sum_{\sigma_n} 1_{\{t_{\sigma_n(1)} \geq \cdots \geq t_{\sigma_n(n)}\}} f_n(t_{\sigma_n(1)}, \cdots, t_{\sigma_n(n)}) \tag{17}$$

with taking summation over all permutations σ_n.

(proof) The assertion can be easily shown:

$$\begin{aligned}
I_n(\hat{f}) &= n! J_n(\hat{f}) \\
&= n! J_n \left(\frac{1}{n!} \sum_{\sigma_n} 1_{\{t_{\sigma_n(1)} \leq \cdots \leq t_{\sigma_n(n)}\}} f(t_{\sigma_n(1)}, \cdots, t_{\sigma_n(n)}) \right) \\
&= \sum_{\sigma_n} \int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} 1_{\{t_{\sigma_n(1)} \leq \cdots \leq t_{\sigma_n(n)}\}} f(t_{\sigma_n(1)}, \cdots, t_{\sigma_n(n)}) dW_{t_n} \cdots dW_{t_2} dW_{t_1} \\
&= \int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(t_1, \cdots, t_n) dW_{t_n} \cdots dW_{t_2} dW_{t_1} = J_n(f). \square
\end{aligned}$$

Finally we introduce and prove the following lemma.

Lemma 2 *Let $A^i_{lt} := \frac{\partial^l X_t^{(\epsilon)}}{\partial \epsilon^l}|_{\epsilon=0}$. Then, it has an expansion with a finite number of iterated Itô integrals as*

$$A^i_{lt} = \sum_{l'=1}^{l} J_{l'}(f^{i,l}_{l'}) \tag{18}$$

with $L^2(\mathbf{T}^{l'})$-function $\{f^{i,l}_{l'}\}$ whose derivation was explained in Section 3.1.

Before the proof of Lemma 2 we state the following lemma.

Lemma 3 *Assume $r = 1$ for simplicity. Then, the n-th Malliavin derivative of $X_t^{(\epsilon),i}$ are given by*

$$\begin{aligned} D_{t_1,\cdots,t_n} X_t^{(\epsilon),i} &= \epsilon \sum_{\eta=1}^{n} \alpha^i_{n-1}(t_\eta; t_1, \cdots, t_{\eta-1}, t_{\eta+1}, \cdots, t_n; \epsilon) \\ &\quad + \epsilon \int_{t_1 \vee \cdots \vee t_n}^{t} \alpha^i_n(s, t_1, \cdots, t_n; \epsilon) dW_s + \int_{t_1 \vee \cdots \vee t_n}^{t} \beta^i_n(s, t_1, \cdots, t_n; \epsilon) ds \end{aligned} \tag{19}$$

for $t \geq t_1 \vee \cdots \vee t_n$ and zero for $t < t_1 \vee \cdots \vee t_n$, where

$$\alpha^i_n(s, t_1, \cdots, t_n; \epsilon) := \sum \partial_{k_1} \cdots \partial_{k_\nu} V^i(X_s^{(\epsilon)}) \prod_{p=1}^{\nu} D_{t(M_p)} X_s^{(\epsilon),k_p}, \tag{20}$$

$$\text{and } \beta^i_n(s, t_1, \cdots, t_n; \epsilon) := \sum \partial_{k_1} \cdots \partial_{k_\nu} V^i_0(X_s^{(\epsilon)}, \epsilon) \prod_{p=1}^{\nu} D_{t(M_p)} X_s^{(\epsilon),k_p} \tag{21}$$

where $t(M_p) = t_{\eta_1}, \cdots, t_{\eta_p}$ for $M_p = \{\eta_1, \cdots, \eta_p; \eta_1 < \cdots < \eta_p\} \subset \{1, \cdots, n\}$, and the sums are taken under the set of all partitions $\{M_p\}_{p=1}^{\nu}$ such that $M_1 \cup \cdots \cup M_\nu = \{1, \cdots, n\}$.

(proof) See pp.123-124 in Nualart [35].□

(proof of Lemma 2) From Lemma 1, it is equivalent to show that A^i_{lT} has a Wiener-Chaos expansion with a finite number of terms as

$$A^i_{lt} = \sum_{l'=1}^{l} I_{l'}(\tilde{f}^{i,l}_{l'}) \tag{22}$$

with $L^2_{sym}(\mathbf{T}^{l'})$-functions $\{\tilde{f}^{i,l}_{l'}\}$. Since $\tilde{f}^{l}_{l'}$ is given by

$$\begin{aligned} \tilde{f}^{i,l}_{l'}(t_1, \cdots, t_{l'}) = \frac{1}{l'!}\mathbf{E}\left[D_{t_1,\cdots,t_{l'}} A^i_{lT}\right] &= \frac{1}{l'!}\mathbf{E}\left[D_{t_1,\cdots,t_{l'}} \frac{\partial^l}{\partial \epsilon^l}\Big|_{\epsilon=0} X_T^{(\epsilon),i}\right] \\ &= \frac{1}{l'!}\mathbf{E}\left[\frac{\partial^l}{\partial \epsilon^l}\Big|_{\epsilon=0} D_{t_1,\cdots,t_{l'}} X_T^{(\epsilon),i}\right] \end{aligned}$$

where the last equality holds due to uniqueness of the asymptotic expansion of $X_T^{(\epsilon),i}$, in order to prove the expansion (22) it is sufficient to see that for any $l' > l$,

$$Y^{i,l}_{l'T} := \frac{\partial^l}{\partial \epsilon^l}\Big|_{\epsilon=0} D_{t_1,\cdots,t_{l'}} X_T^{(\epsilon),i}$$

is equal to zero, which will be proved by induction.

First, it is obvious that this statement holds with $l=0$, because $X_t^{(\epsilon)}$ becomes deterministic as $\epsilon \downarrow 0$.

Second, from Lemma 3, for $l \geq 1$ we have

$$
\begin{aligned}
Y_{l't}^{i,l} &= l \sum_{\eta=1}^{l'} \frac{\partial^{l-1}}{\partial \epsilon^{l-1}}\Big|_{\epsilon=0} \big(\alpha_{l'-1}^{i}(t_\eta; t_1, \cdots, t_{\eta-1}, t_{\eta+1}, \cdots, t_{l'}; \epsilon)\big) \qquad (23) \\
&\quad + l \int_{t_1 \vee \cdots \vee t_{l'}}^{t} \frac{\partial^{l-1}}{\partial \epsilon^{l-1}}\Big|_{\epsilon=0} \big(\alpha_{l'}^{i}(s, t_1, \cdots, t_{l'}; \epsilon)\big) dW_s + \int_{t_1 \vee \cdots \vee t_{l'}}^{t} \frac{\partial^{l}}{\partial \epsilon^{l}}\Big|_{\epsilon=0} \big(\beta_{l'}^{i}(s, t_1, \cdots, t_{l'}; \epsilon)\big) ds.
\end{aligned}
$$

for $t \geq t_1 \vee \cdots \vee t_n$ and $Y_{l't}^{i,l} = 0$ for $t < t_1 \vee \cdots \vee t_n$.

First and second terms of the right hand side of (23) is summation of the terms (for the second term whose integrands are)

$$
\frac{\partial^{l_0}}{\partial \epsilon^{l_0}} \left(\partial_{k_1} \cdots \partial_{k_\nu} V^i(X_s^{(\epsilon)})\right) \prod_{p=1}^{\nu} \frac{\partial^{l_p}}{\partial \epsilon^{l_p}}\Big|_{\epsilon=0} D_{t(M_p)} X_s^{(\epsilon),k_p}
$$

with the conditions $\sum_{p=0}^{\nu} l_p = l-1$ and $\sum_{p=1}^{\nu} \#(M_p) = k$, $k = l'-1$ for the first term and $k = l'$ for the second term. Thus, since $k > l-1$ in both cases, for at least one p we have $l_p < \#(M_p)$. Then, all of these terms will vanish as $\epsilon \downarrow 0$ by the assumption of induction(note that $l_p \leq l-1$).

For the third terms, almost the same result is obtained except that $\int \partial_j V_0^i(X_s^{(\epsilon)}, \epsilon) Y_{l's}^{j,l} ds (j = 1, \cdots, d)$ remains.

As a consequence, all terms except for

$$
\sum_{j=1}^{d} \int_{t_1 \vee \cdots \vee t_{l'}}^{t} \partial_j V_0^i(X_s^{(0)}, 0) Y_{l's}^{j,l} ds
$$

are equal to zero. Thus we have a trivial linear equation

$$
Y_{l't}^{l} = \int_{t_1 \vee \cdots \vee t_{l'}}^{t} \partial V_0(X_s^{(0)}, 0) Y_{l's}^{l} ds \qquad (24)
$$

whose solution is given by $Y_{l't}^{l} \equiv (0, \cdots, 0)'$ where $Y_{l't}^{l} = (Y_{l't}^{1,l}, \cdots, Y_{l't}^{d,l})'$. □

3.3 Useful formulas

Finally, we here list up some formulas of conditional expectations often used in asymptotic expansions. Let $q_i : [0,T] \mapsto \mathbf{R}^m, i = 1,2,3,4,5,6,7$ are non-random functions and we define Σ as

$$
\Sigma = \int_0^T q_{1v}' q_{1v} dv,
$$

where z' is the transpose of z. We assume that $0 < \Sigma < \infty$ and integrability in the following formulas.

Before the list of formulas, we define a notation of iterated integrations for convenience;

$$
F_n(f_1, \cdots, f_n) := \int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} f_1(t_1) \cdots f_n(t_n) dt_n \cdots dt_1, \ n \geq 1. \qquad (25)
$$

1.

$$\begin{aligned}&\mathbf{E}\left[\int_0^T q'_{2t}dW_t|\int_0^T q'_{1v}dW_v = x\right]\\ =\quad& F_1(q'_2q_1)\frac{H_1(x;\Sigma)}{\Sigma}\end{aligned}$$

2.

$$\begin{aligned}&\mathbf{E}\left[\int_0^T\int_0^t q'_{2u}dW_u q'_{3t}dW_t|\int_0^T q'_{1v}dW_v = x\right]\\ =\quad& F_2(q'_2q_1, q'_3q_1)\frac{H_2(x;\Sigma)}{\Sigma^2}\end{aligned}$$

3.

$$\begin{aligned}&\mathbf{E}\left[\left(\int_0^T q'_{2u}dW_u\right)\left(\int_0^T q'_{3s}dW_s\right)|\int_0^T q'_{1v}dW_v = x\right]\\ =\quad& \left(F_1(q'_2q_1)\times F_1(q'_3q_1)\right)\frac{H_2(x;\Sigma)}{\Sigma^2} + F_1(q'_2q_3)\end{aligned}$$

4.

$$\begin{aligned}&\mathbf{E}\left[\int_0^T\int_0^t\int_0^s q'_{2u}dW_u q'_{3s}dW_s q'_{4t}dW_t|\int_0^T q'_{1v}dW_v = x\right]\\ =\quad& F_3(q'_2q_1, q'_3q_1, q'_4q_1)\frac{H_3(x;\Sigma)}{\Sigma^3}\end{aligned}$$

5.

$$\begin{aligned}&\mathbf{E}\left[\int_0^T\left(\int_0^t q'_{2u}dW_u\right)\left(\int_0^t q'_{3s}dW_s\right)q'_{4t}dW_t|\int_0^T q'_{1v}dW_v = x\right]\\ =\quad& \left(F_3(q'_2q_1, q'_3q_1, q'_4q_1;T) + F_3(q'_3q_1, q'_2q_1, q'_4q_1)\right)\frac{H_3(x;\Sigma)}{\Sigma^3} + F_2(q'_2q_3, q'_4q_1)\frac{H_1(x;\Sigma)}{\Sigma}\end{aligned}$$

6.

$$\begin{aligned}&\mathbf{E}\left[\left(\int_0^T\int_0^t q'_{2s}dW_s q'_{3t}dW_t\right)\left(\int_0^T q'_{4u}dW_u\right)|\int_0^T q'_{1v}dW_v = x\right]\\ =\quad& \left(F_2(q'_2q_1, q'_3q_1)\times F_1(q'_4q_1)\right)\frac{H_3(x;\Sigma)}{\Sigma^3} + \left(F_2(q'_2q_4, q'_3q_1) + F_2(q'_2q_1, q'_3q_4)\right)\frac{H_1(x;\Sigma)}{\Sigma}\end{aligned}$$

7.

$$\begin{aligned}&\mathbf{E}\left[\left(\int_0^T\int_0^t q'_{2s}dW_s q'_{3t}dW_t\right)\left(\int_0^T\int_0^r q'_{4u}dW_u q'_{5r}dW_r\right)|\int_0^T q'_{1v}dW_v = x\right]\\ =\quad& \left(F_2(q'_2q_1, q'_3q_1)\times F_2(q'_4q_1, q'_5q_1)\right)\frac{H_4(x;\Sigma)}{\Sigma^4}\\ &+\left\{\left(F_3(q'_2q_4, q'_5q_1, q'_3q_1) + F_3(q'_2q_1, q'_3q_4, q'_5q_1) + F_3(q'_2q_1, q'_4q_1, q'_3q_5)\right)\right.\\ &+\left.\left(F_3(q'_2q_4, q'_3q_1, q'_5q_1) + F_3(q'_4q_1, q'_2q_5, q'_3q_1) + F_3(q'_4q_1, q'_2q_1, q'_3q_5)\right)\right\}\frac{H_2(x;\Sigma)}{\Sigma^2}\\ &+F_2(q'_2q_4, q'_3q_5)\\ =:\quad& \tilde{F}_4^7(q_1,q_2,q_3,q_4,q_5)\frac{H_4(x;\Sigma)}{\Sigma^4} + \tilde{F}_2^7(q_1,q_2,q_3,q_4,q_5)\frac{H_2(x;\Sigma)}{\Sigma^2} + \tilde{F}_0^7(q_1,q_2,q_3,q_4,q_5)\end{aligned}$$

8.

$$\begin{aligned}
&\mathbf{E}\left[\left(\int_0^T q'_{2t}dW_t\right)\left(\int_0^T q'_{3s}dW_s\right)\left(\int_0^T\int_0^r q'_{4u}dW_u q'_{5r}dW_r\right) \mid \int_0^T q'_{1v}dW_v = x\right] \\
= \quad & \left(\tilde{F}_4^7(q_1,q_2,q_3,q_4,q_5)+\tilde{F}_4^7(q_1,q_3,q_2,q_4,q_5)\right)\frac{H_4(x;\Sigma)}{\Sigma^4} \\
& \left(\tilde{F}_2^7(q_1,q_2,q_3,q_4,q_5)+\tilde{F}_2^7(q_1,q_3,q_2,q_4,q_5)+F_1(q'_2q_3)\times F_2(q'_4q_1,q'_5q_1)\right)\frac{H_2(x;\Sigma)}{\Sigma^2} \\
& +\left(\tilde{F}_2^7(q_1,q_2,q_3,q_4,q_5)+\tilde{F}_2^7(q_1,q_3,q_2,q_4,q_5)\right)
\end{aligned}$$

9.

$$\begin{aligned}
&\mathbf{E}\left[\int_0^T\int_0^t\int_0^s\int_0^u q'_{2r}dW_r q'_{3u}dW_u q'_{4s}dW_s q'_{5t}dW_t \mid \int_0^T q'_{1v}dW_v = x\right] \\
= \quad & F_4(q'_2q_1,q'_3q_1,q'_4q_1,q'_5q_1)\frac{H_4(x;\Sigma)}{\Sigma^4}
\end{aligned}$$

10.

$$\begin{aligned}
&\mathbf{E}\left[\int_0^T\int_0^t\left(\int_0^u q'_{2s}dW_s\right)\left(\int_0^u q'_{3r}dW_r\right) q'_{4u}dW_u q'_{5t}dW_t \mid \int_0^T q'_{1v}dW_v = x\right] \\
= \quad & \left(F_4(q'_2q_1,q'_3q_1,q'_4q_1,q'_5q_1)+F_4(q'_3q_1,q'_2q_1,q'_4q_1,q'_5q_1)\right)\frac{H_4(x;\Sigma)}{\Sigma^4} \\
& +F_3(q'_2q_3,q'_4q_1,q'_5q_1)\frac{H_2(x;\Sigma)}{\Sigma^2}
\end{aligned}$$

11.

$$\begin{aligned}
&\mathbf{E}\left[\int_0^T\left(\int_0^t\int_0^u q'_{2r}dW_r q'_{3u}dW_u\right)\left(\int_0^t q'_{4u}dW_u\right) q'_{5t}dW_t \mid \int_0^T q'_{1v}dW_v = x\right] \\
= \quad & \left(F_4(q'_2q_1,q'_3q_1,q'_4q_1,q'_5q_1)+F_4(q'_2q_1,q'_4q_1,q'_3q_1,q'_5q_1)+F_4(q'_4q_1,q'_2q_1,q'_3q_1,q'_5q_1)\right)\frac{H_4(x;\Sigma)}{\Sigma^4} \\
& +\left(F_3(q'_2q_4,q'_3q_1,q'_5q_1)+F_3(q'_2q_1,q'_3q_4,q'_5q_1)\right)\frac{H_2(x;\Sigma)}{\Sigma^2} \\
=: \quad & \tilde{F}_4^{11}(q_1,q_2,q_3,q_4,q_5)\frac{H_4(x;\Sigma)}{\Sigma^4}+\tilde{F}_2^{11}(q_1,q_2,q_3,q_4,q_5)\frac{H_2(x;\Sigma)}{\Sigma^2}
\end{aligned}$$

12.

$$\begin{aligned}
&\mathbf{E}\left[\int_0^T\left(\int_0^t q'_{2s}dW_s\right)\left(\int_0^t q'_{3r}dW_r\right)\left(\int_0^t q'_{4u}dW_u\right) q'_{5t}dW_t \mid \int_0^T q'_{1v}dW_v = x\right] \\
= \quad & \left(\tilde{F}_4^{11}(q_1,q_2,q_3,q_4,q_5)+\tilde{F}_4^{11}(q_1,q_3,q_2,q_4,q_5)\right)\frac{H_4(x;\Sigma)}{\Sigma^4} \\
& +\left\{\tilde{F}_2^{11}(q_1,q_2,q_3,q_4,q_5)+\tilde{F}_2^{11}(q_1,q_3,q_2,q_4,q_5)+\left(F_3(q'_2q_3,q'_4q_1,q'_5q_1)+F_3(q'_4q_1,q'_2q_3,q'_5q_1)\right)\right\}\frac{H_2(x;\Sigma)}{\Sigma^2}
\end{aligned}$$

13.

$$
\begin{aligned}
&\mathbf{E}\left[\left(\int_0^T\int_0^t q'_{2s}dW_s q'_{3t}dW_t\right)\left(\int_0^T\int_0^t\int_0^u q'_{4s}dW_s q'_{5u}dW_u q'_{6t}dW_t\right) \mid \int_0^T q'_{1v}dW_v = x\right]\\
= \quad & \left(F_2(q'_2q_1, q'_3q_1)\times F_3(q'_4q_1, q'_5q_1, q'_6q_1)\right)\frac{H_5(x;\Sigma)}{\Sigma^5}\\
&+\Bigg\{\left(F_4(q'_4q_1, q'_5q_1, q'_2q_1, q'_3q_6) + F_4(q'_4q_1, q'_2q_1, q'_5q_1, q'_3q_6) + F_4(q'_2q_1, q'_4q_1, q'_5q_1, q'_3q_6)\right)\\
&+\ \left(F_4(q'_4q_1, q'_3q_5, q'_2q_1, q'_6q_1) + F_4(q'_4q_1, q'_2q_1, q'_3q_5, q'_3q_1) + F_4(q'_2q_1, q'_4q_1, q'_3q_5, q'_3q_1)\right)\\
&+\ \Big(F_4(q'_2q_1, q'_3q_4, q'_5q_1, q'_6q_1) + F_4(q'_4q_1, q'_2q_5, q'_3q_1, q'_6q_1)\\
&\qquad\qquad + F_4(q'_2q_4, q'_3q_1, q'_5q_1, q'_6q_1) + F_4(q'_2q_4, q'_5q_1, q'_3q_1, q'_6q_1)\Big)\\
&+\ \left(F_4(q'_4q_1, q'_5q_1, q'_2q_6, q'_3q_1) + F_4(q'_4q_1, q'_2q_5, q'_6q_1, q'_3q_1) + F_4(q'_2q_4, q'_5q_1, q'_6q_1, q'_3q_1)\right)\Bigg\}\frac{H_3(x;\Sigma)}{\Sigma^3}\\
&+\left(F_3(q'_4q_1, q'_2q_5, q'_3q_6) + F_3(q'_2q_4, q'_5q_1, q'_3q_6) + F_3(q'_2q_4, q'_3q_5, q'_6q_1)\right)\frac{H_1(x;\Sigma)}{\Sigma}\\
=: \quad & \tilde F_5^{13}(q_1,q_2,q_3,q_4,q_5,q_6)\frac{H_5(x;\Sigma)}{\Sigma^5} + \tilde F_3^{13}(q_1,q_2,q_3,q_4,q_5,q_6)\frac{H_3(x;\Sigma)}{\Sigma^3} + \tilde F_1^{13}(q_1,q_2,q_3,q_4,q_5,q_6)\frac{H_1(x;\Sigma)}{\Sigma}
\end{aligned}
$$

14.

$$
\begin{aligned}
&\mathbf{E}\left[\left(\int_0^T\int_0^t q'_{2s}dW_s q'_{3t}dW_t\right)\left(\int_0^T\left(\int_0^t q'_{4s}dW_s\right)\left(\int_0^t q'_{5u}dW_u\right) q'_{6t}dW_t\right) \mid \int_0^T q'_{1v}dW_v = x\right]\\
= \quad & \left(\tilde F_5^{13}(q_1,q_2,q_3,q_4,q_5,q_6) + \tilde F_5^{13}(q_1,q_2,q_3,q_5,q_4,q_6)\right)\frac{H_5(x;\Sigma)}{\Sigma^5}\\
&+\left(\tilde F_3^{13}(q_1,q_2,q_3,q_4,q_5,q_6) + \tilde F_3^{13}(q_1,q_2,q_3,q_5,q_4,q_6) + F_2(q'_2q_1, q'_3q_1)\times F_2(q'_4q_5, q'_6q_1)\right)\frac{H_3(x;\Sigma)}{\Sigma^3}\\
&+\Big(\tilde F_1^{13}(q_1,q_2,q_3,q_4,q_5,q_6) + \tilde F_1^{13}(q_1,q_2,q_3,q_5,q_4,q_6)\\
&\qquad + F_3(q'_4q_5, q'_2q_6, q'_3q_1) + F_3(q'_2q_1, q'_4q_5, q'_3q_6) + F_3(q'_4q_5, q'_2q_1, q'_3q_6)\Big)\frac{H_1(x;\Sigma)}{\Sigma}
\end{aligned}
$$

15.

$$
\begin{aligned}
&\mathbf{E}\left[\left(\int_0^T\int_0^t q'_{2u}dW_s q'_{3t}dW_t\right)\left(\int_0^T\int_0^t q'_{4u}dW_u q'_{5t}dW_r\right)\left(\int_0^T\int_0^t q'_{6u}dW_u q'_{7t}dW_r\right) \mid \int_0^T q'_{1v}dW_v = x\right]\\
= \quad & \left(F_2(q'_2q_1, q'_3q_1)\times F_2(q'_4q_1, q'_5q_1)\times F_2(q'_6q_1, q'_7q_1)\right)\frac{H_6(x;\Sigma)}{\Sigma^6}\\
&+\left(\tilde F_{4^*}^{14}(q_1,q_2,q_3,q_4,q_5,q_6,q_7) + \tilde F_{4^*}^{14}(q_1,q_4,q_5,q_6,q_7,q_2,q_3) + \tilde F_{4^*}^{14}(q_1,q_6,q_7,q_2,q_3,q_4,q_5)\right)\frac{H_4(x;\Sigma)}{\Sigma^4}\\
&+\left(\tilde F_{2^*}^{14}(q_1,q_2,q_3,q_4,q_5,q_6,q_7) + \tilde F_{2^*}^{14}(q_1,q_4,q_5,q_6,q_7,q_2,q_3) + \tilde F_{2^*}^{14}(q_1,q_6,q_7,q_2,q_3,q_4,q_5)\right)\frac{H_2(x;\Sigma)}{\Sigma^2}\\
&+\Big(F_3(q'_2q_6, q'_4q_7, q'_3q_5) + F_3(q'_4q_6, q'_2q_7, q'_3q_5)\\
&\qquad + F_3(q'_2q_4, q'_3q_6, q'_5q_7) + F_3(q'_2q_6, q'_3q_4, q'_5q_7) + F_3(q'_4q_6, q'_2q_5, q'_3q_7) + F_3(q'_2q_4, q'_5q_6, q'_3q_7)\Big)
\end{aligned}
$$

where

$$
\begin{aligned}
&\tilde{F}^{14}_{4^*}(q_1,q_2,q_3,q_4,q_5,q_6,q_7)\\
:=\ &\Big(F_3(q_4'q_6,q_5'q_1,q_7'q_1)+F_3(q_4'q_6,q_7'q_1,q_5'q_1)+F_3(q_6'q_1,q_4'q_7,q_5'q_1)\\
&\quad +F_3(q_4'q_1,q_5'q_6,q_7'q_1)+F_3(q_4'q_1,q_6'q_1,q_5'q_7)+F_3(q_6'q_1,q_4'q_1,q_5'q_7)\Big)\times F_2(q_2'q_1,q_3'q_1),\\
&\tilde{F}^{14}_{2^*}(q_1,q_2,q_3,q_4,q_5,q_6,q_7)\\
:=\ &F_2(q_2'q_1,q_3'q_1)\times F_2(q_4'q_6,q_5'q_7)\\
&\quad +\Big(F_4(q_2'q_4,q_5'q_1,q_3'q_6,q_7'q_1)+F_4(q_2'q_4,q_3'q_6,q_5'q_1,q_7'q_1)+F_4(q_2'q_4,q_3'q_6,q_7'q_1,q_5'q_1)\\
&\qquad +F_4(q_2'q_6,q_7'q_1,q_3'q_4,q_5'q_1)+F_4(q_2'q_6,q_3'q_4,q_7'q_1,q_5'q_1)+F_4(q_2'q_6,q_3'q_4,q_5'q_1,q_7'q_1)\Big)\\
&\quad +\Big(F_4(q_4'q_1,q_2'q_5,q_3'q_6,q_7'q_1)+F_4(q_4'q_1,q_2'q_6,q_3'q_5,q_7'q_1)+F_4(q_4'q_1,q_2'q_6,q_7'q_1,q_3'q_5)\\
&\qquad +F_4(q_2'q_6,q_4'q_1,q_3'q_5,q_7'q_1)+F_4(q_2'q_6,q_4'q_1,q_7'q_1,q_3'q_5)+F_4(q_2'q_6,q_7'q_1,q_4'q_1,q_3'q_5)\Big)\\
&\quad +\Big(F_4(q_6'q_1,q_2'q_7,q_3'q_4,q_5'q_1)+F_4(q_6'q_1,q_2'q_4,q_3'q_7,q_5'q_1)+F_4(q_6'q_1,q_2'q_4,q_5'q_1,q_3'q_7)\\
&\qquad +F_4(q_2'q_4,q_6'q_1,q_3'q_7,q_5'q_1)+F_4(q_2'q_4,q_6'q_1,q_5'q_1,q_3'q_7)+F_4(q_2'q_4,q_5'q_1,q_6'q_1,q_3'q_7)\Big)\\
&\quad +\Big(F_4(q_4'q_1,q_2'q_5,q_6'q_1,q_3'q_7)+F_4(q_4'q_1,q_6'q_1,q_2'q_5,q_3'q_7)+F_4(q_4'q_1,q_6'q_1,q_2'q_7,q_3'q_5)\\
&\qquad +F_4(q_6'q_1,q_4'q_1,q_2'q_5,q_3'q_7)+F_4(q_6'q_1,q_4'q_1,q_4'q_1,q_3'q_5)+F_4(q_6'q_1,q_2'q_7,q_2'q_7,q_3'q_5)\Big)
\end{aligned}
$$

4 New Computational Scheme

In this section we propose a new computational scheme in asymptotic expansion which is alternative to the method described in the previous section. To compute conditional expectations in the right hand side of (7), we use the following lemma which can be derived from the property of Hermite polynomials.

Lemma 4 *Let (Ω, F, P) be a probability space. Suppose that $X \in L^2(\Omega)$ and Z is a random variable with Gaussian distribution with mean 0 and variance Σ. Then, the conditional expectation $E[X|Z]$ has following expansion in $L^2(\Omega)$:*

$$E[X|Z] = \sum_{n=0}^{\infty} a_n H_n(Z;\Sigma) \tag{26}$$

where $H_n(z;\Sigma)$ is the Hermite polynomial of degree n which is defined as

$$H_n(z;\Sigma) = (-\Sigma)^n e^{z^2/2\Sigma} \frac{d^n}{dx^n} e^{-z^2/2\Sigma}$$

and coefficients a_n are given by

$$a_n = \frac{1}{(i\Sigma)^n} \frac{d^n}{d\xi^n}\bigg|_{\xi=0} \left\{ e^{\frac{\xi^2}{2}\Sigma} \mathbf{E}[e^{i\xi Z} X] \right\}. \tag{27}$$

(proof) Since the Hermite polynomials $\{H_n(z;\Sigma)\}$ is the orthogonal basis of $L^2(\mathbf{R},\mu)$ where μ is the Gaussian measure on $\mathbf{R}$ with mean 0 and variance

Σ, and $E[X|Z=z] \in L^2(\mathbf{R},\mu)$, we have the following unique expansion of $E[X|Z=z]$ in $L^2(\mathbf{R},\mu)$:

$$E[X|Z=z] = \sum_{n=0}^{\infty} a_n H_n(z;\Sigma)$$

and also we have

$$E[X|Z] = \sum_{n=0}^{\infty} a_n H_n(Z;\Sigma)$$

in $L^2(\Omega)$. And note that

$$e^{i\xi Z} = e^{-\frac{\xi^2}{2}\Sigma} \sum_{n=0}^{\infty} \frac{H_n(Z;\Sigma)}{n!}(i\xi)^n.$$

Then,

$$\begin{aligned} e^{\frac{\xi^2}{2}\Sigma}\mathbf{E}[e^{i\xi Z}X] &= e^{\frac{\xi^2}{2}\Sigma}\mathbf{E}[e^{i\xi Z}\mathbf{E}[X|Z]] \\ &= \mathbf{E}[\sum_{m=0}^{\infty}\frac{H_m(Z;\Sigma)}{m!}(i\xi)^m \sum_{n=0}^{\infty} a_n H_n(Z;\Sigma)] \\ &= \sum_{n=0}^{\infty} a_n (i\Sigma)^n \xi^n. \end{aligned}$$

Comparing to the coefficients of the Taylor series of $e^{\frac{\xi^2}{2}\Sigma}\mathbf{E}[e^{i\xi Z}X]$ around 0 with respect to ξ, we see that a_n can be written as (27).□

Recall $\hat{g}_{1T}$ is defined as

$$\hat{g}_{1T} = (\partial g(X_T^{(0)}))' \int_0^T [Y_T Y_t^{-1} V(X_t^{(0)})] dW_t = g_{1T} - C$$

where

$$C = (\partial g(X_T^{(0)}))' \int_0^T Y_T Y_t^{-1} \partial_\epsilon V_0(X_t^{(0)}, 0) dt,$$

and define

$$Z_T^{\langle\xi\rangle} = \exp\{i\xi\hat{g}_{1T} + \frac{\xi^2}{2}\Sigma_T\}.$$

Note that, as see in later, in the asymptotic expansion scheme, the infinite series expansion of $\mathbf{E}[X^{j,m,k}|Z_T^{\langle\xi\rangle}]$ can be deduced to a finite sum up to the $(j+m)$th term by the analogy with the previous section.

Then, from Lemma 4 and (7), we have the following expression of $\mathbf{E}[\Phi(G^{(\epsilon)})]$:

$$\begin{aligned} \mathbf{E}[\Phi(G^{(\epsilon)})] &= \sum_{j=0}^{N}\epsilon^j \sum_{m=0}^{j}\frac{1}{m!}\int_{\mathbf{R}}\Phi(x)\sum_{k\in K_{j,m}} C^{j,m,k}(-1)^m \frac{\partial^m}{\partial x^m}\{\mathbf{E}\left[X^{j,m,k}\middle|\,\hat{g}_{1T}=x-C\right] f_{g_{1T}}(x)\}dx + o(\epsilon^N) \\ &= \sum_{j=0}^{N}\epsilon^j \sum_{m=0}^{j}\frac{1}{m!}\int_{\mathbf{R}}\Phi(x)\sum_{k\in K_{j,m}} C^{j,m,k}(-1)^m \frac{\partial^m}{\partial x^m}\{\sum_{l=0}^{j+m} a_l^{j,m,k} H_l(x-C;\Sigma_T) f_{g_{1T}}(x)\}dx + o(\epsilon^N) \end{aligned}$$

where

$$a_l^{j,m,k} = \frac{1}{(i\Sigma_T)^l}\left.\frac{d^l}{d\xi^l}\right|_{\xi=0}\left\{\mathbf{E}[X^{j,m,k} Z_T^{\langle\xi\rangle}]\right\}.$$

In particular, let Φ be the delta function at $x \in \mathbf{R}$, δ_x, we obtain the asymptotic expansion of density of $G^{(\epsilon)}$:

$$
\begin{aligned}
f_{G^{(\epsilon)}}(x) &= \mathbf{E}[\delta_x(G^{(\epsilon)})] \\
&= \sum_{j=0}^{N} \epsilon^j \sum_{m=0}^{j} \sum_{k \in K_{j,m}} \sum_{l=0}^{j+m} \frac{a_l^{j,m,k} C^{j,m,k}}{m!} (-1)^m \frac{\partial^m}{\partial x^m} \{H_l(x-C;\Sigma_T) f_{g_{1T}}(x)\} + o(\epsilon^N).
\end{aligned} \tag{28}
$$

4.1 Asymptotic Expansion of Density Function

In this subsection, we propose a new computational method for the asymptotic expansion of the density function (28). In particular, we show that coefficients in the expansion is obtained through a system of ordinary differential equations that is solved easily, and derive a concrete expression of the expansion up to ϵ^3-order.

First, we write down the equation (28) more explicitly up to ϵ^3-order:

$$
\begin{aligned}
f_{G^{(\epsilon)}}(x) &= a_0^{0,0,(0)} H_0(x-C;\Sigma_T) f_{g_{1T}}(x) \\
&+\epsilon \left\{ \sum_{l=0}^{2} a_l^{1,1,(1)} (-1) \frac{\partial}{\partial x} \{H_l(x-C;\Sigma_T) f_{g_{1T}}(x)\} \right\} \\
&+\epsilon^2 \left\{ \sum_{l=0}^{3} a_l^{2,1,(0,1)} (-1) \frac{\partial}{\partial x} \{H_l(x-C;\Sigma_T) f_{g_{1T}}(x)\} \right. \\
&\left. + \frac{1}{2} \sum_{l=0}^{4} a_l^{2,2,(2,0)} \frac{\partial^2}{\partial x^2} \{H_l(x-C;\Sigma_T) f_{g_{1T}}(x)\} \right\} \\
&+\epsilon^3 \left\{ \sum_{l=0}^{4} a_l^{3,1,(0,0,1)} (-1) \frac{\partial}{\partial x} \{H_l(x-C;\Sigma_T) f_{g_{1T}}(x)\} \right. \\
&+ \frac{1}{2} \sum_{l=0}^{5} a_l^{3,2,(1,1,0)} \frac{\partial^2}{\partial x^2} \{H_l(x-C;\Sigma_T) f_{g_{1T}}(x)\} \\
&\left. + \frac{1}{6} \sum_{l=0}^{6} a_l^{3,3,(3,0,0)} (-1) \frac{\partial^3}{\partial x^3} \{H_l(x-C;\Sigma_T) f_{g_{1T}}(x)\} \right\} \\
&+o(\epsilon^3),
\end{aligned}
$$

where coefficients $a_l^{j,m,k}$ are given by

$$
\begin{aligned}
a_l^{0,0,(0)} &= \frac{1}{(i\Sigma_T)^l}\frac{d^l}{d\xi^l}\bigg|_{\xi=0}\left\{\mathbf{E}[Z_T^{\langle\xi\rangle}]\right\} \\
a_l^{1,1,(1)} &= \frac{1}{(i\Sigma_T)^l}\frac{d^l}{d\xi^l}\bigg|_{\xi=0}\left\{\mathbf{E}[g_{2T}Z_T^{\langle\xi\rangle}]\right\} \\
a_l^{2,1,(0,1)} &= \frac{1}{(i\Sigma_T)^l}\frac{d^l}{d\xi^l}\bigg|_{\xi=0}\left\{\mathbf{E}[g_{3T}Z_T^{\langle\xi\rangle}]\right\} \\
a_l^{2,2,(2,0)} &= \frac{1}{(i\Sigma_T)^l}\frac{d^l}{d\xi^l}\bigg|_{\xi=0}\left\{\mathbf{E}[g_{2T}^2Z_T^{\langle\xi\rangle}]\right\} \\
a_l^{3,1,(0,0,1)} &= \frac{1}{(i\Sigma_T)^l}\frac{d^l}{d\xi^l}\bigg|_{\xi=0}\left\{\mathbf{E}[g_{4T}Z_T^{\langle\xi\rangle}]\right\} \\
a_l^{3,2,(1,1,0)} &= \frac{1}{(i\Sigma_T)^l}\frac{d^l}{d\xi^l}\bigg|_{\xi=0}\left\{\mathbf{E}[g_{2T}g_{3T}Z_T^{\langle\xi\rangle}]\right\} \\
a_l^{3,3,(3,0,0)} &= \frac{1}{(i\Sigma_T)^l}\frac{d^l}{d\xi^l}\bigg|_{\xi=0}\left\{\mathbf{E}[g_{2T}^3Z_T^{\langle\xi\rangle}]\right\}.
\end{aligned}
\tag{29}
$$

Since $E[Z_T^{\langle\xi\rangle}]=1$, we have $a_0^{0,0,(0)}=1$ and $a_l^{0,0,(0)}=0$ for $l\geq 1$. The other expectations above are expressed in terms of A_{kT} and $Z_T^{\langle\xi\rangle}$ as follows:

$$
\begin{aligned}
\mathbf{E}[g_{2T}Z_T^{\langle\xi\rangle}] &= \frac{1}{2}\sum_{i,j=1}^{d}\partial_i\partial_j g(X_T^{(0)})\mathbf{E}[A_{1T}^iA_{1T}^jZ_T^{\langle\xi\rangle}] + \frac{1}{2}\sum_{i=1}^{d}\partial_i g(X_T^{(0)})\mathbf{E}[A_{2T}^iZ_T^{\langle\xi\rangle}] \\
\mathbf{E}[g_{3T}Z_T^{\langle\xi\rangle}] &= \frac{1}{6}\sum_{i,j,k=1}^{d}\partial_i\partial_j\partial_k g(X_T^{(0)})\mathbf{E}[A_{1T}^iA_{1T}^jA_{1T}^kZ_T^{\langle\xi\rangle}] + \frac{1}{2}\sum_{i,j=1}^{d}\partial_i\partial_j g(X_T^{(0)})\mathbf{E}[A_{2T}^iA_{1T}^jZ_T^{\langle\xi\rangle}] \\
&\quad +\frac{1}{6}\sum_{i=1}^{d}\partial_i g(X_T^{(0)})\mathbf{E}[A_{3T}^iZ_T^{\langle\xi\rangle}], \\
\mathbf{E}[g_{2T}^2Z_T^{\langle\xi\rangle}] &= \frac{1}{4}\sum_{i,j,k,l=1}^{d}\partial_i\partial_j g(X_T^{(0)})\partial_k\partial_l g(X_T^{(0)})\mathbf{E}[A_{1T}^iA_{1T}^jA_{1T}^kA_{1T}^lZ_T^{\langle\xi\rangle}] \\
&\quad +\frac{1}{2}\sum_{i,j,k=1}^{d}\partial_i\partial_j g(X_T^{(0)})\partial_i g(X_T^{(0)})\mathbf{E}[A_{1T}^iA_{1T}^jA_{2T}^kZ_T^{\langle\xi\rangle}] \\
&\quad +\frac{1}{4}\sum_{i,j=1}^{d}\partial_i g(X_T^{(0)})\partial_j g(X_T^{(0)})\mathbf{E}[A_{2T}^iA_{2T}^jZ_T^{\langle\xi\rangle}] \\
\mathbf{E}[g_{4T}Z_T^{\langle\xi\rangle}] &= \frac{1}{24}\sum_{i,j,k,l=1}^{d}\partial_i\partial_j\partial_k\partial_l g(X_T^{(0)})\mathbf{E}[A_{1T}^iA_{1T}^jA_{1T}^kA_{1T}^lZ_T^{\langle\xi\rangle}] \\
&\quad +\frac{1}{4}\sum_{i,j,k=1}^{d}\partial_i\partial_j\partial_k g(X_T^{(0)})\mathbf{E}[A_{2T}^iA_{1T}^jA_{1T}^kZ_T^{\langle\xi\rangle}] \\
&\quad +\frac{1}{2}\sum_{i,j=1}^{d}\partial_i\partial_j g(X_T^{(0)})\mathbf{E}[A_{2T}^iA_{2T}^jZ_T^{\langle\xi\rangle}] + \frac{1}{2}\sum_{i,j=1}^{d}\partial_i\partial_j g(X_T^{(0)})\mathbf{E}[A_{3T}^iA_{1T}^jZ_T^{\langle\xi\rangle}] \\
&\quad +\frac{1}{6}\sum_{i=1}^{d}\partial_i g(X_T^{(0)})\mathbf{E}[A_{4T}^iZ_T^{\langle\xi\rangle}], \\
\mathbf{E}[g_{2T}g_{3T}Z_T^{\langle\xi\rangle}] &= \frac{1}{12}\sum_{i,j,k,l,m=1}^{d}\partial_i\partial_j g(X_T^{(0)})\partial_k\partial_l\partial_m g(X_T^{(0)})\mathbf{E}[A_{1T}^iA_{1T}^jA_{1T}^kA_{1T}^lA_{1T}^mZ_T^{\langle\xi\rangle}] \\
&\quad +\frac{1}{12}\sum_{i,j,k,l=1}^{d}\Big\{\partial_i g(X_T^{(0)})\partial_j\partial_k\partial_l g(X_T^{(0)}) + 3\partial_i\partial_j g(X_T^{(0)})\partial_k\partial_l g(X_T^{(0)})\Big\}\mathbf{E}[A_{1T}^iA_{1T}^jA_{1T}^kA_{2T}^lZ_T^{\langle\xi\rangle}] \\
&\quad +\frac{1}{4}\sum_{i,j,k=1}^{d}\partial_k g(X_T^{(0)})\partial_i\partial_j g(X_T^{(0)})\mathbf{E}[A_{1T}^iA_{2T}^jA_{2T}^kZ_T^{\langle\xi\rangle}] \\
&\quad +\frac{1}{12}\sum_{i,j,k=1}^{d}\partial_i\partial_j g(X_T^{(0)})\partial_k g(X_T^{(0)})\mathbf{E}[A_{1T}^iA_{1T}^jA_{3T}^kZ_T^{\langle\xi\rangle}] \\
&\quad +\frac{1}{12}\sum_{i,j=1}^{d}\partial_i g(X_T^{(0)})\partial_j g(X_T^{(0)})\mathbf{E}[A_{2T}^iA_{3T}^jZ_T^{\langle\xi\rangle}], \\
\mathbf{E}[g_{2T}^3Z_T^{\langle\xi\rangle}] &= \frac{1}{8}\sum_{i,j,k,l,m,n=1}^{d}\partial_i\partial_j g(X_T^{(0)})\partial_k\partial_l g(X_T^{(0)})\partial_m\partial_n g(X_T^{(0)})\mathbf{E}[A_{1T}^iA_{1T}^jA_{1T}^kA_{1T}^lA_{1T}^mA_{1T}^nZ_T^{\langle\xi\rangle}] \\
&\quad +\frac{3}{8}\sum_{i,j,k,l,m=1}^{d}\partial_i\partial_j g(X_T^{(0)})\partial_k\partial_l g(X_T^{(0)})\partial_m g(X_T^{(0)})\mathbf{E}[A_{1T}^iA_{1T}^jA_{1T}^kA_{1T}^lA_{2T}^mZ_T^{\langle\xi\rangle}] \\
&\quad +\frac{3}{8}\sum_{i,j,k,l,m=1}^{d}\partial_i\partial_j g(X_T^{(0)})\partial_k g(X_T^{(0)})\partial_l g(X_T^{(0)})\mathbf{E}[A_{1T}^iA_{1T}^jA_{2T}^kA_{2T}^lZ_T^{\langle\xi\rangle}] \\
&\quad +\frac{1}{8}\sum_{i,j,k,l,m=1}^{d}\partial_i g(X_T^{(0)})\partial_j g(X_T^{(0)})\partial_k g(X_T^{(0)})\mathbf{E}[A_{2T}^iA_{2T}^jA_{2T}^kZ_T^{\langle\xi\rangle}]
\end{aligned}
$$

where A_{1t} is given by (2), and $A_{kt} := \frac{\partial^k X_t^{(\epsilon)}}{\partial \epsilon^k}|_{\epsilon=0}$, $k = 2, 3, 4$ are expressed as follows:

$$
\begin{aligned}
A_{2t} &= \int_0^t Y_t Y_u^{-1} \Bigg(\sum_{j,k=1}^d \partial_j \partial_k V_0(X_u^{(0)}, 0) A_{1u}^j A_{1u}^k du + 2 \sum_{j=1}^d \partial_\epsilon \partial_j V_0(X_u^{(0)}, 0) A_{1u}^j du \\
&\quad + \partial_\epsilon^2 V_0(X_u^{(0)}, 0) du + 2 \sum_{j=1}^d \partial_j V(X_u^{(0)}) A_{1u}^j dW_u \Bigg), \\
A_{3t} &= \int_0^t Y_t Y_u^{-1} \Bigg(\sum_{j,k,l=1}^d \partial_j \partial_k \partial_l V_0(X_u^{(0)}, 0) A_{1u}^j A_{1u}^k A_{1u}^l du + 3 \sum_{j,k=1}^d \partial_j \partial_k V_0(X_u^{(0)}, 0) A_{1u}^j A_{2u}^k du \\
&\quad + 3 \sum_{j,k=1}^d \partial_j \partial_k \partial_\epsilon V_0(X_u^{(0)}, 0) A_{1u}^j A_{1u}^k du + 3 \sum_{j=1}^d \partial_j \partial_\epsilon V_0(X_u^{(0)}, 0) A_{2u}^j du \\
&\quad + 3 \sum_{j=1}^d \partial_j \partial_\epsilon^2 V_0(X_u^{(0)}, 0) A_{1u}^j du + \partial_\epsilon^3 V_0(X_u^{(0)}, 0) du \\
&\quad + 3 \sum_{j,k=1}^d \partial_j \partial_k V(X_u^{(0)}) A_{1u}^j A_{1u}^k dW_u + 3 \sum_{j=1}^d \partial_j V(X_u^{(0)}) A_{2u}^j dW_u \Bigg), \\
A_{4t} &= \int_0^t Y_t Y_u^{-1} \Bigg(\sum_{j,k,l,m=1}^d \partial_j \partial_k \partial_l \partial_m V_0(X_u^{(0)}, 0) A_{1u}^j A_{1u}^k A_{1u}^l A_{1u}^m du \\
&\quad + 4 \sum_{j,k,l=1}^d \partial_j \partial_k \partial_l V_0(X_u^{(0)}, 0) A_{1u}^j A_{1u}^k A_{2u}^l du + 3 \sum_{j,k=1}^d \partial_j \partial_k V_0(X_u^{(0)}, 0) A_{2u}^j A_{2u}^k du \\
&\quad + 4 \sum_{j,k=1}^d \partial_j \partial_k V(X_u^{(0)}) A_{1u}^j A_{3u}^k du + 4 \sum_{j,k,l=1}^d \partial_j \partial_k \partial_l \partial_\epsilon V_0(X_u^{(0)}, 0) A_{1u}^j A_{1u}^k A_{1u}^l du \\
&\quad + 6 \sum_{j,k=1}^d \partial_j \partial_k \partial_\epsilon V_0(X_u^{(0)}, 0) A_{1u}^j A_{2u}^k du + 4 \sum_{j=1}^d \partial_j \partial_\epsilon V_0(X_u^{(0)}, 0) A_{3u}^j du \\
&\quad + 6 \sum_{j,k=1}^d \partial_j \partial_k \partial_\epsilon^2 V_0(X_u^{(0)}, 0) A_{1u}^j A_{1u}^k du + 6 \sum_{j=1}^d \partial_j \partial_\epsilon^2 V_0(X_u^{(0)}, 0) A_{2u}^j du \\
&\quad + 4 \sum_{j=1}^d \partial_j \partial_\epsilon^3 V_0(X_u^{(0)}, 0) A_{1u}^j du + \partial_\epsilon^4 V_0(X_u^{(0)}, 0) du \\
&\quad + 4 \sum_{j,k,l=1}^d \partial_j \partial_k \partial_l V(X_u^{(0)}) A_{1u}^j A_{1u}^k A_{1u}^l dW_u + 12 \sum_{j,k=1}^d \partial_j \partial_k V(X_u^{(0)}) A_{1u}^j A_{2u}^k dW_u \\
&\quad + 4 \sum_{j=1}^d \partial_j V(X_u^{(0)}) A_{3u}^j dW_u \Bigg).
\end{aligned}
$$

Note that each $A_{kt}^i (i = 1, \cdots, d, k = 1, 2, 3, 4)$ has all finite moments due to a *grading* structure. For the detail of the following definition and theorem, see pp.45-47 in Bichteler, Gravereaux and Jacod [1].

Definition 1 *A grading of $\mathbf{R}^d$ is a decomposition $\mathbf{R}^d = \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_q}$ with $d = d_1 + \cdots + d_q$. The coordinates of a point in $\mathbf{R}^d$ are always arranged in an increasing order along the subspace $\mathbf{R}^{d_i}$, and we set $M_0 = 0$ and $M_l = d_1 + \cdots + d_l$ for $1 \leq l \leq q$. We say that a mapping F on $\mathbf{R}^d$ is graded according* *to the grading* $\mathbf{R}^d = \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_q}$ *if* $F^i(s)$ *depends upon only through the coordinates* $(s^k)_{1\le k\le M_r}$ *when* $M_{r-1} \le i \le M_r$.

Theorem 1 *Consider the stochastic differential equation of the form*

$$dS_t = \mu(S_t,t)dt + \sigma(S_t,t)dW_t;\ S_0 = s_0 \in \mathbf{R}^d \tag{30}$$

where coefficients $\mu : \mathbf{R}^d \times \mathbf{R}^+ \to \mathbf{R}^d$ *and* $\sigma : \mathbf{R}^d \times \mathbf{R}^+ \to \mathbf{R}^d \otimes \mathbf{R}^{d'}$ *have a Lipschitz lower triangular structure, and are graded according to* $\mathbf{R}^d = \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_q}$ *with respect to* S. *Moreover for* $F(s,t) = \mu(s,t)$ *or* $\sigma(s,t)$, *we assume* F *is differentiable in* s *in* $\mathbf{R}^d$ *and*

1. $|F(0,t)| \le Z_t$
2. $|D_sF(s,t)| \le \hat{Z}_t(1+|y|^\theta)$
3. $|\partial_j F^i(s,t)| \le \zeta$ *if* $M_{r-1} \le i \le M_r$ *for some* $r \le q$

where $\zeta, \theta \ge 0$ *are constants, and* Z, $\hat{Z}$ *are predictable processes such that* $||Z||_p$ *and* $||\hat{Z}||_p$ *are finite for all* $p \ge 1$. *Then (30) have a unique solution* S, *and for every* $p \ge 1$ *there are constants* c_p *and* γ_p *depending only upon* $(\zeta, \theta, \{||\hat{Z}||_r\}_{r\ge 1})$, *such that*

$$||S_T||_p \le c_p(s_0 + ||Z||_{\gamma_p}).$$

Applying Theorem 1 to the system of stochastic differential equations consists of $A^i_{kt} (i = 1, \cdots, d, k = 1,2,3,4)$ and any products of them, we obtain the following lemma.

Lemma 5 *Each coefficient of the expansion* $A^i_{kt} (i = 1, \cdots, d, k = 1,2,3,4)$ *has all finite moments.*

(proof) Consider the system of stochastic differential equations which $A^1_1, \cdots, A^d_1$, $A^1_1A^1_1, \cdots, A^d_1A^d_1$, $A^1_2, \cdots, A^d_2, \cdots$ follow, then it is easily shown that the coefficients of the equation have a grading structure and satisfy the conditions in Theorem 1. Hence the coefficients A^i_{kt} have all finite moments.□

Here, we redefine $\hat{g}_1 = \{\hat{g}_{1t}; t \in \mathbf{R}^+\}$ and $Z^{\langle\xi\rangle} = \{Z^{\langle\xi\rangle}_t; t \in \mathbf{R}^+\}$ as the stochastic processes

$$\hat{g}_{1t} = \int_0^t \hat{V}(X^{(0)}_u, u)dW_u$$

and

$$Z^{\langle\xi\rangle}_t = \exp\{i\xi\hat{g}_{1t} + \frac{\xi^2}{2}\Sigma_t\},$$

respectively where

$$\hat{V}(x,t) = (\partial g(X^{(0)}_T))'[Y_TY_t^{-1}V(x)].$$

We define $\eta^i_{1,1}$, $\eta^i_{2,1}$, $\eta^{i,k}_{2,2}$, $\eta^i_{3,1}$, $\eta^{i,k}_{3,2}$, $\eta^{i,k}_{3,3}$, $\eta^i_{4,1}$, $\eta^{i,k}_{4,2,1}$, $\eta^{i,k}_{4,2,2}$, $\eta^{i,k}_{5,2}$, $\eta^{i,k}_{5,3}$, and

$\eta_{6,3}^{i,k,l}$ as

$$\begin{aligned}
&\eta_{1,1}^{i}(t) := E[A_{1t}^{i}Z_t^{\langle\xi\rangle}], \quad \eta_{2,1}^{i}(t) := E[A_{2t}^{i}Z_t^{\langle\xi\rangle}], \quad \eta_{2,2}^{i,k}(t) := E[A_{1t}^{i}A_{1t}^{k}Z_t^{\langle\xi\rangle}],\\
&\eta_{3,1}^{i}(t) := E[A_{3t}^{i}Z_t^{\langle\xi\rangle}], \quad \eta_{3,2}^{i,k}(t) := E[A_{1t}^{i}A_{2t}^{k}Z_t^{\langle\xi\rangle}], \quad \eta_{3,3}^{i,k,l}(t) := E[A_{1t}^{i}A_{1t}^{k}A_{1t}^{l}Z_t^{\langle\xi\rangle}],\\
&\eta_{4,1}^{i}(t) := E[A_{4t}^{i}Z_t^{\langle\xi\rangle}], \quad \eta_{4,2,1}^{i,k}(t) := E[A_{1t}^{i}A_{3t}^{k}Z_t^{\langle\xi\rangle}], \quad \eta_{4,2,2}^{i,k}(t) := E[A_{2t}^{i}A_{2t}^{k}Z_t^{\langle\xi\rangle}],\\
&\eta_{4,3}^{i,k,l}(t) := E[A_{1t}^{i}A_{1t}^{k}A_{2t}^{l}Z_t], \quad \eta_{4,4}^{i,k,l,m}(t) := E[A_{1t}^{i}A_{1t}^{k}A_{1t}^{l}A_{1t}^{m}Z_t^{\langle\xi\rangle}],\\
&\eta_{5,2}^{i,k}(t) := E[A_{2t}^{i}A_{3t}^{k}Z_t^{\langle\xi\rangle}], \quad \eta_{5,3,1}^{i,k,l}(t) := E[A_{1t}^{i}A_{1t}^{k}A_{3t}^{l}Z_t^{\langle\xi\rangle}], \quad \eta_{5,3,2}^{i,k,l}(t) := E[A_{1t}^{i}A_{2t}^{k}A_{2t}^{l}Z_t^{\langle\xi\rangle}],\\
&\eta_{5,4}^{i,k,l,m}(t) := E[A_{1t}^{i}A_{1t}^{k}A_{1t}^{l}A_{2t}^{m}Z_t^{\langle\xi\rangle}], \quad \eta_{5,5}^{i,k,l,m,n}(t) := E[A_{1t}^{i}A_{1t}^{k}A_{1t}^{l}A_{1t}^{m}A_{1t}^{n}Z_t^{\langle\xi\rangle}],\\
&\eta_{6,3}^{i,k,l}(t) := E[A_{2t}^{i}A_{2t}^{k}A_{2t}^{l}Z_t^{\langle\xi\rangle}], \quad \eta_{6,4}^{i,k,l,m}(t) := E[A_{1t}^{i}A_{1t}^{k}A_{2t}^{l}A_{2t}^{m}Z_t^{\langle\xi\rangle}],\\
&\eta_{6,5}^{i,k,l,m,n}(t) := E[A_{1t}^{i}A_{1t}^{k}A_{1t}^{l}A_{1t}^{m}A_{2t}^{n}Z_t^{\langle\xi\rangle}], \quad \eta_{6,5}^{i,k,l,m,n,o}(t) := E[A_{1t}^{i}A_{1t}^{k}A_{1t}^{l}A_{1t}^{m}A_{1t}^{n}A_{1t}^{o}Z_t^{\langle\xi\rangle}].
\end{aligned} \tag{31}$$

We derive the system of ordinary differential equations of η.

In the followings, for simplicity, we assume that V_0 doesn't depend on ϵ, and write $V_0(x,\epsilon)$ as $V_0(x)$.

Consider the evaluation of $\eta_{2,1}^{i}(T) = E[A_{2T}^{i}Z_T^{\langle\xi\rangle}]$ which appears in the ϵ-order. Applying Ito's formula to $A_{2t}^{i}Z_t^{\langle\xi\rangle}$, we have

$$\begin{aligned}
d(A_{2t}^{i}Z_t^{\langle\xi\rangle}) &= A_{2t}^{i}dZ_t^{\langle\xi\rangle} + Z_t^{\langle\xi\rangle}dA_{2t}^{i} + dA_{2t}^{i}dZ_t^{\langle\xi\rangle}\\
&= \Bigg\{2(i\xi)\sum_{i'=1}^{d}A_{1t}^{i'}Z_t^{\langle\xi\rangle}\hat{V}(X_t^{(0)},t)\partial_{i'}V^{i}(X_t^{(0)})' + \sum_{i'=1}^{d}A_{2t}^{i'}Z_t^{\langle\xi\rangle}\partial_{i'}V_0^{i}(X_t^{(0)})\\
&\quad + \sum_{i'=1}^{d}\sum_{k'=1}^{d}A_{1t}^{i'}A_{1t}^{k'}Z_t^{\langle\xi\rangle}\partial_{i'}\partial_{k'}V_0^{i}(X_t^{(0)})\Bigg\}dt\\
&\quad + \Bigg\{(i\xi)A_{2t}^{i}Z_t^{\langle\xi\rangle}\hat{V}(X_t^{(0)},t) + 2\sum_{i'=1}^{d}A_{1t}^{i'}Z_t^{\langle\xi\rangle}\partial_{i'}V^{i}(X_t^{(0)})\Bigg\}dW_t.
\end{aligned}$$

Since the second and third terms are martingales, taking the expectation on both sides, we have the following ordinary differential equation of $\eta_{2,1}^{i}$:

$$\begin{aligned}
\frac{d}{dt}\eta_{2,1}^{i}(t) &= 2(i\xi)\sum_{i'=1}^{d}\eta_{1,1}^{i'}(t)\hat{V}(X_t^{(0)},t)\partial_{i'}V^{i}(X_t^{(0)})'\\
&\quad + \sum_{i'=1}^{d}\eta_{2,1}^{i'}(t)\partial_{i'}V_0^{i}(X_t^{(0)}) + \sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{2,2}^{i',k'}(t)\partial_{i'}\partial_{k'}V_0^{i}(X_t^{(0)}).
\end{aligned}$$

Here, $\eta_{1,1}^{i}(i=1,\cdots,d)$ appearing in the right hand side of above ODE are evaluated in the similar manner:

$$\begin{aligned}
d(A_{1t}^{i}Z_t^{\langle\xi\rangle}) &= A_{1t}^{i}dZ_t^{\langle\xi\rangle} + Z_t^{\langle\xi\rangle}dA_{1t}^{i} + dA_{1t}^{i}dZ_t^{\langle\xi\rangle}\\
&= \Bigg\{(i\xi)Z_t^{\langle\xi\rangle}\hat{V}(X_t^{(0)},t)V^{i}(X_t^{(0)})' + \sum_{i'=1}^{d}A_{1t}^{i'}Z_t^{\langle\xi\rangle}\partial_{i'}V_0^{i}(X_t^{(0)})\Bigg\}dt\\
&\quad + \Big\{(i\xi)A_{1t}^{i}Z_t^{\langle\xi\rangle}\hat{V}(X_t^{(0)},t) + Z_t^{\langle\xi\rangle}V^{i}(X_t^{(0)})\Big\}dW_t,
\end{aligned}$$

hence, we have

$$\frac{d}{dt}\eta_{1,1}^{i}(t) = (i\xi)\hat{V}(X_t^{(0)},t)V^{i}(X_t^{(0)})' + \sum_{i'=1}^{d}\eta_{1,1}^{i'}(t)\partial_{i'}V_0^{i}(X_t^{(0)}).$$

$\eta^{i,k}_{2,2}$ and other higher order terms can be evaluated in the same way.

The key observation is that each ODE does not involve any higher order terms, and only lower or the same order terms appear in the right hand side of the ODE. So, one can easily solve (analytically or numerically) the system of ODEs and evaluate expectations.

Proposition 2 *For $\eta_{j,m,k}$ defined in (31), the following system of ordinary differential equations is hold:*

$$\frac{d}{dt}\eta^{i}_{1,1}(t) = (i\xi)\hat{V}(X^{(0)}_t,t)V^i(X^{(0)}_t)' + \sum_{i'=1}^{d}\eta^{i'}_{1,1}(t)\partial_{i'}V^i_0(X^{(0)}_t)$$

$$\begin{aligned}
\frac{d}{dt}\eta^{i}_{2,1}(t) &= 2(i\xi)\sum_{i'=1}^{d}\eta^{i'}_{1,1}(t)\hat{V}(X^{(0)}_t,t)\partial_{i'}V^i(X^{(0)}_t)' \\
&\quad + \sum_{i'=1}^{d}\eta^{i'}_{2,1}(t)\partial_{i'}V^i_0(X^{(0)}_t) + \sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta^{i',k'}_{2,2}(t)\partial_{i'}\partial_{k'}V^i_0(X^{(0)}_t) \\
\frac{d}{dt}\eta^{i,k}_{2,2}(t) &= (i\xi)\left\{\eta^{k}_{1,1}(t)\hat{V}(X^{(0)}_t,t)V^i(X^{(0)}_t)' + \eta^{i}_{1,1}(t)\hat{V}(X^{(0)}_t,t)V^k(X^{(0)}_t)'\right\} \\
&\quad + V^i(X^{(0)}_t)V^k(X^{(0)}_t)' + \sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta^{i',k'}_{2,2}(t)\partial_{i'}V^i_0(X^{(0)}_t)\partial_{k'}V^k_0(X^{(0)}_t)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\eta^{i}_{3,1}(t) &= (i\xi)\Big\{3\sum_{i'=1}^{d}\eta^{i'}_{2,1}(t)\hat{V}(X^{(0)}_t,t)\partial_{i'}V^i(X^{(0)}_t)' \\
&\quad + 3\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta^{i',k'}_{2,2}(t)\hat{V}(X^{(0)}_t,t)\partial_{i'}\partial_{k'}V^i(X^{(0)}_t)'\Big\} \\
&\quad + \sum_{i'=1}^{d}\eta^{i'}_{3,1}(t)\partial_{i'}V^i_0(X^{(0)}_t) + 3\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta^{i',k'}_{3,2}(t)\partial_{i'}\partial_{k'}V^i_0(X^{(0)}_t) \\
&\quad + \sum_{i'=1}^{d}\sum_{k'=1}^{d}\sum_{l'=1}^{d}\eta^{i',k',l'}_{3,3}(t)\partial_{i'}\partial_{k'}\partial_{l'}V^i_0(X^{(0)}_t) \\
\frac{d}{dt}\eta^{i,k}_{3,2}(t) &= (i\xi)\Big\{\eta^{k}_{2,1}(t)\hat{V}(X^{(0)}_t,t)V^i(X^{(0)}_t)' + 2\sum_{i'=1}^{d}\eta^{i,i'}_{2,2}(t)\hat{V}(X^{(0)}_t,t)\partial_{i'}V^k(X^{(0)}_t)'\Big\} \\
&\quad + 2\sum_{i'=1}^{d}\eta^{i'}_{1,1}(t)V^i(X^{(0)}_t)\partial_{i'}V^k(X^{(0)}_t)' \\
&\quad + \sum_{i'=1}^{d}\eta^{i',k}_{3,2}(t)\partial_{i'}V^i_0(X^{(0)}_t) + \sum_{k'=1}^{d}\eta^{i,k'}_{3,2}(t)\partial_{k'}V^k_0(X^{(0)}_t) \\
&\quad + \sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta^{i,i',k'}_{3,3}(t)\partial_{i'}\partial_{k'}V^k_0(X^{(0)}_t) \\
\frac{d}{dt}\eta^{i,k,l}_{3,3}(t) &= (i\xi)\Big\{\eta^{k,l}_{2,2}(t)\hat{V}(X^{(0)}_t,t)V^i(X^{(0)}_t)' + \eta^{i,l}_{2,2}(t)\hat{V}(X^{(0)}_t,t)V^k(X^{(0)}_t)' + \eta^{i,k}_{2,2}(t)\hat{V}(X^{(0)}_t,t)V^l(X^{(0)}_t)'\Big\} \\
&\quad + \eta^{i}_{1,1}(t)V^k(X^{(0)}_t)V^l(X^{(0)}_t)' + \eta^{k}_{1,1}(t)V^i(X^{(0)}_t)V^l(X^{(0)}_t)' + \eta^{l}_{1,1}(t)V^i(X^{(0)}_t)V^k(X^{(0)}_t)' \\
&\quad + \sum_{i'=1}^{d}\eta^{i',k,l}_{3,3}(t)\partial_{i'}V^i_0(X^{(0)}_t) + \sum_{k'=1}^{d}\eta^{i,k',l}_{3,3}(t)\partial_{k'}V^k_0(X^{(0)}_t) + \sum_{l'=1}^{d}\eta^{i,k,l'}_{3,3}(t)\partial_{l'}V^l_0(X^{(0)}_t)
\end{aligned}$$

$$
\begin{aligned}
\frac{d}{dt}\eta^{i}_{4,1}(t) &= (i\xi)\Big\{4\sum_{i'=1}^{d}\eta^{i'}_{3,1}(t)\hat{V}(X^{(0)}_t,t)\partial_{i'}V^i(X^{(0)}_t)' + 12\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta^{i',k'}_{3,2}(t)\hat{V}(X^{(0)}_t,t)\partial_{i'}\partial_{k'}V^i(X^{(0)}_t)' \\
&\quad +4\sum_{i'=1}^{d}\sum_{k'=1}^{d}\sum_{l'=1}^{d}\eta^{i',k',l'}_{3,3}(t)\hat{V}(X^{(0)}_t,t)\partial_{i'}\partial_{k'}\partial_{l'}V^i(X^{(0)}_t)'\Big\} \\
&\quad +\sum_{i'=1}^{d}\eta^{i'}_{4,1}(t)\partial_{i'}V^i_0(X^{(0)}_t) + 4\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta^{i',k'}_{4,2,1}(t)\partial_{i'}\partial_{k'}V^i_0(X^{(0)}_t) \\
&\quad +3\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta^{i',k'}_{4,2,2}(t)\partial_{i'}\partial_{k'}V^i_0(X^{(0)}_t) + 4\sum_{i'=1}^{d}\sum_{k'=1}^{d}\sum_{l'=1}^{d}\eta^{i',k',l'}_{4,3}(t)\partial_{i'}\partial_{k'}\partial_{l'}V^i_0(X^{(0)}_t) \\
&\quad +\sum_{i'=1}^{d}\sum_{k'=1}^{d}\sum_{l'=1}^{d}\sum_{m'=1}^{d}\eta^{i',k',l',m'}_{4,4}(t)\partial_{i'}\partial_{k'}\partial_{l'}\partial_{m'}V^i_0(X^{(0)}_t) \\
\frac{d}{dt}\eta^{i,k}_{4,2,1}(t) &= (i\xi)\Big\{\eta^{k}_{3,1}(t)\hat{V}(X^{(0)}_t,t)V^i(X^{(0)}_t)' + 3\sum_{i'=1}^{d}\eta^{i,i'}_{3,2}(t)\hat{V}(X^{(0)}_t,t)\partial_{i'}V^k(X^{(0)}_t)' \\
&\quad +3\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta^{i,i',k'}_{3,3}(t)\hat{V}(X^{(0)}_t,t)\partial_{i'}\partial_{k'}V^k(X^{(0)}_t)'\Big\} \\
&\quad +3\sum_{i'=1}^{d}\eta^{i'}_{2,1}(t)V^i(X^{(0)}_t)\partial_{i'}V^k(X^{(0)}_t)' + 3\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta^{i',k'}_{2,2}(t)V^i(X^{(0)}_t)\partial_{i'}\partial_{k'}V^k(X^{(0)}_t)' \\
&\quad +\sum_{i'=1}^{d}\eta^{i',k}_{4,2,1}(t)\partial_{i'}V^i_0(X^{(0)}_t) + \sum_{k'=1}^{d}\eta^{i,k'}_{4,2,1}(t)\partial_{k'}V^k_0(X^{(0)}_t) \\
&\quad +3\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta^{i,i',k'}_{4,3}(t)\partial_{i'}\partial_{k'}V^k_0(X^{(0)}_t) + \sum_{i'=1}^{d}\sum_{k'=1}^{d}\sum_{l'=1}^{d}\eta^{i,i',k',l'}_{4,4}(t)\partial_{i'}\partial_{k'}\partial_{l'}V^k_0(X^{(0)}_t) \\
\frac{d}{dt}\eta^{i,k}_{4,2,2}(t) &= (i\xi)\Big\{2\sum_{i'=1}^{d}\eta^{i',k}_{3,2}(t)\hat{V}(X^{(0)}_t,t)\partial_{i'}V^i(X^{(0)}_t)' + 2\sum_{i'=1}^{d}\eta^{i',i}_{3,2}(t)\hat{V}(X^{(0)}_t,t)\partial_{i'}V^k(X^{(0)}_t)'\Big\} \\
&\quad +4\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta^{i',k'}_{2,2}(t)\partial_{i'}V^i(X^{(0)}_t)\partial_{k'}V^k(X^{(0)}_t)' \\
&\quad +\sum_{i'=1}^{d}\eta^{i',k}_{4,2,2}(t)\partial_{i'}V^i_0(X^{(0)}_t) + \sum_{k'=1}^{d}\eta^{i,k'}_{4,2,2}(t)\partial_{k'}V^k_0(X^{(0)}_t) \\
&\quad +\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta^{i',k',k}_{4,3}(t)\partial_{i'}\partial_{k'}V^i_0(X^{(0)}_t) + \sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta^{i',k',i}_{4,3}(t)\partial_{i'}\partial_{k'}V^k_0(X^{(0)}_t) \\
\frac{d}{dt}\eta^{i,k,l}_{4,3}(t) &= (i\xi)\Big\{\eta^{k,l}_{3,2}(t)\hat{V}(X^{(0)}_t,t)V^i(X^{(0)}_t)' + \eta^{i,l}_{3,2}(t)\hat{V}(X^{(0)}_t,t)V^k(X^{(0)}_t)' \\
&\quad +2\sum_{i'=1}^{d}\eta^{i,k,i'}_{3,3}(t)\hat{V}(X^{(0)}_t,t)\partial_{i'}V^l(X^{(0)}_t)'\Big\} \\
&\quad +\eta^{l}_{2,1}(t)V^i(X^{(0)}_t)V^k(X^{(0)}_t)' \\
&\quad +2\sum_{i'=1}^{d}\eta^{i,i'}_{2,2}(t)V^k(X^{(0)}_t)\partial_{i'}V^l(X^{(0)}_t)' + 2\sum_{i'=1}^{d}\eta^{k,i'}_{2,2}(t)V^i(X^{(0)}_t)\partial_{i'}V^l(X^{(0)}_t)' \\
&\quad +\sum_{i'=1}^{d}\eta^{i',k,l}_{4,3}(t)\partial_{i'}V^i_0(X^{(0)}_t) + \sum_{k'=1}^{d}\eta^{i,k',l}_{4,3}(t)\partial_{k'}V^k_0(X^{(0)}_t) + \sum_{l'=1}^{d}\eta^{i,k,l'}_{4,3}(t)\partial_{l'}V^l_0(X^{(0)}_t) \\
&\quad +\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta^{i,k,i',k'}_{4,4}(t)\partial_{i'}\partial_{k'}V^l_0(X^{(0)}_t)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\eta_{4,4}^{i,k,l,m}(t) &= (i\xi)\Big\{\eta_{3,3}^{k,l,m}(t)\hat{V}(X_t^{(0)},t)V^i(X_t^{(0)})' + \eta_{3,3}^{i,l,m}(t)\hat{V}(X_t^{(0)},t)V^k(X_t^{(0)})' \\
&+\eta_{3,3}^{i,k,m}(t)\hat{V}(X_t^{(0)},t)V^l(X_t^{(0)})' + \eta_{3,3}^{i,k,l}(t)\hat{V}(X_t^{(0)},t)V^m(X_t^{(0)})'\Big\} \\
&+\eta_{2,2}^{i,k}(t)V^l(X_t^{(0)})V^m(X_t^{(0)})' + \eta_{2,2}^{i,l}(t)V^k(X_t^{(0)})V^m(X_t^{(0)})' + \eta_{2,2}^{i,m}(t)V^k(X_t^{(0)})V^l(X_t^{(0)})' \\
&+\eta_{2,2}^{k,l}(t)V^i(X_t^{(0)})V^m(X_t^{(0)})' + \eta_{2,2}^{k,m}(t)V^i(X_t^{(0)})V^l(X_t^{(0)})' + \eta_{2,2}^{l,m}(t)V^i(X_t^{(0)})V^k(X_t^{(0)})' \\
&+\sum_{i'=1}^{d}\eta_{4,4}^{i',k,l,m}(t)\partial_{i'}V_0^i(X_t^{(0)}) + \sum_{k'=1}^{d}\eta_{4,4}^{i,k',l,m}(t)\partial_{k'}V_0^k(X_t^{(0)}) \\
&+\sum_{l'=1}^{d}\eta_{4,4}^{i,k,l',m}(t)\partial_{l'}V_0^l(X_t^{(0)}) + \sum_{m'=1}^{d}\eta_{4,4}^{i,k,l,m'}(t)\partial_{m'}V_0^m(X_t^{(0)})
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\eta_{5,2}^{i,k}(t) &= (i\xi)\Big\{2\sum_{i'=1}^{d}\eta_{4,2,1}^{i',k}(t)\hat{V}(X_t^{(0)},t)\partial_{i'}V^i(X_t^{(0)})' + 3\sum_{i'=1}^{d}\eta_{4,2,2}^{i,i'}(t)\hat{V}(X_t^{(0)},t)\partial_{i'}V^k(X_t^{(0)})' \\
&+3\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{4,3}^{i',k',i}(t)\hat{V}(X_t^{(0)},t)\partial_{i'}\partial_{k'}V^k(X_t^{(0)})'\Big\} \\
&+6\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{3,2}^{i',k'}(t)\partial_{i'}V^i(X_t^{(0)})\partial_{k'}V^k(X_t^{(0)})' \\
&+6\sum_{i'=1}^{d}\sum_{k'=1}^{d}\sum_{l'=1}^{d}\eta_{3,3}^{i',k',l'}(t)\partial_{i'}V^i(X_t^{(0)})\partial_{k'}\partial_{l'}V^k(X_t^{(0)})' \\
&+\sum_{i'=1}^{d}\eta_{5,2}^{i',k}(t)\partial_{i'}V_0^i(X_t^{(0)}) + \sum_{k'=1}^{d}\eta_{5,2}^{i,k'}(t)\partial_{k'}V_0^k(X_t^{(0)}) \\
&+\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{5,3,1}^{i',k',k}(t)\partial_{i'}\partial_{k'}V_0^i(X_t^{(0)}) + 3\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{5,3,2}^{i',k',i}(t)\partial_{i'}\partial_{k'}V_0^k(X_t^{(0)}) \\
&+\sum_{i'=1}^{d}\sum_{k'=1}^{d}\sum_{l'=1}^{d}\eta_{5,4}^{i',k',l',i}(t)\partial_{i'}\partial_{k'}\partial_{l'}V_0^k(X_t^{(0)})
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\eta_{5,3,1}^{i,k,l}(t) &= (i\xi)\Big\{\eta_{4,2,1}^{k,l}(t)\hat{V}(X_t^{(0)},t)V^i(X_t^{(0)})' + \eta_{4,2,1}^{i,l}(t)\hat{V}(X_t^{(0)},t)V^k(X_t^{(0)})' \\
&+3\sum_{i'=1}^{d}\eta_{4,3}^{i,k,i'}(t)\hat{V}(X_t^{(0)},t)\partial_{i'}V^l(X_t^{(0)})' + 3\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{4,4}^{i,k,i',k'}(t)\hat{V}(X_t^{(0)},t)\partial_{i'}\partial_{k'}V^l(X_t^{(0)})'\Big\} \\
&+\eta_{3,1}(t)V^i(X_t^{(0)})'V^k(X_t^{(0)}) \\
&+3\sum_{i'=1}^{d}\eta_{3,2}^{k,i'}(t)V^i(X_t^{(0)})\partial_{i'}V^l(X_t^{(0)})' + 3\sum_{i'=1}^{d}\eta_{3,2}^{i,i'}(t)V^k(X_t^{(0)})\partial_{i'}V^l(X_t^{(0)})' \\
&+3\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{3,3}^{k,i',k'}(t)V^i(X_t^{(0)})\partial_{i'}\partial_{k'}V^l(X_t^{(0)})' + 3\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{3,3}^{i,i',k'}(t)V^k(X_t^{(0)})\partial_{i'}\partial_{k'}V^l(X_t^{(0)})' \\
&+\sum_{i'=1}^{d}\eta_{5,3,1}^{i',k,l}(t)\partial_{i'}V_0^i(X_t^{(0)}) + \sum_{k'=1}^{d}\eta_{5,3,1}^{i,k',l}(t)\partial_{k'}V_0^k(X_t^{(0)}) + \sum_{l'=1}^{d}\eta_{5,3,1}^{i,k,l'}(t)\partial_{l'}V_0^l(X_t^{(0)}) \\
&+3\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{5,4}^{i,k,i',k'}(t)\partial_{i'}\partial_{k'}V_0^l(X_t^{(0)}) + \sum_{i'=1}^{d}\sum_{k'=1}^{d}\sum_{l'=1}^{d}\eta_{5,5}^{i,k,i',k',l'}(t)\partial_{i'}\partial_{k'}\partial_{l'}V_0^l(X_t^{(0)})
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\eta^{i,k,l}_{5,3,2}(t) &= (i\xi)\Big\{\eta^{k,l}_{4,2,2}(t)\hat{V}(X^{(0)}_t,t)V^i(X^{(0)}_t)' \\
&+2\sum_{i'=1}^{d}\eta^{i',i,l}_{4,3}(t)\hat{V}(X^{(0)}_t,t)\partial_{i'}V^k(X^{(0)}_t)' + 2\sum_{i'=1}^{d}\eta^{i',i,k}_{4,3}(t)\hat{V}(X^{(0)}_t,t)\partial_{i'}V^l(X^{(0)}_t)'\Big\} \\
&+2\sum_{i'=1}^{d}\eta^{i',k}_{3,2}(t)V^i(X^{(0)}_t)\partial_{i'}V^l(X^{(0)}_t)' + 2\sum_{i'=1}^{d}\eta^{i',l}_{3,2}(t)V^i(X^{(0)}_t)\partial_{i'}V^k(X^{(0)}_t)' \\
&+4\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta^{i,i',k'}_{3,3}(t)\partial_{i'}V^k(X^{(0)}_t)\partial_{k'}V^l(X^{(0)}_t)' \\
&+\sum_{i'=1}^{d}\eta^{i',k,l}_{5,3,2}(t)\partial_{i'}V^i_0(X^{(0)}_t) + \sum_{k'=1}^{d}\eta^{i,k',l}_{5,3,2}(t)\partial_{k'}V^k_0(X^{(0)}_t) + \sum_{l'=1}^{d}\eta^{i,k,l'}_{5,3,2}(t)\partial_{l'}V^l_0(X^{(0)}_t) \\
&+\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta^{i,i',k',l}_{5,4}(t)\partial_{i'}\partial_{k'}V^k_0(X^{(0)}_t) + \sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta^{i,i',k',k}_{5,4}(t)\partial_{i'}\partial_{k'}V^l_0(X^{(0)}_t) \\
\frac{d}{dt}\eta^{i,k,l,m}_{5,4}(t) &= (i\xi)\Big\{\eta^{k,l,m}_{4,3}(t)\hat{V}(X^{(0)}_t,t)V^i(X^{(0)}_t)' + \eta^{l,i,m}_{4,3}(t)\hat{V}(X^{(0)}_t,t)V^k(X^{(0)}_t)' \\
&+\eta^{i,k,m}_{4,3}(t)\hat{V}(X^{(0)}_t,t)V^l(X^{(0)}_t)' + 2\sum_{i'=1}^{d}\eta^{i,k,l,i'}_{4,4}(t)\hat{V}(X^{(0)}_t,t)\partial_{i'}V^m(X^{(0)}_t)'\Big\} \\
&+\eta^{l,m}_{3,2}(t)V^i(X^{(0)}_t)V^k(X^{(0)}_t)' + \eta^{k,m}_{3,2}(t)V^i(X^{(0)}_t)V^l(X^{(0)}_t)' + \eta^{i,m}_{3,2}(t)V^k(X^{(0)}_t)V^l(X^{(0)}_t)' \\
&+2\sum_{l'=1}^{d}\eta^{k,l,i'}_{3,3}(t)V^i(X^{(0)}_t)\partial_{i'}V^m(X^{(0)}_t)' + 2\sum_{l'=1}^{d}\eta^{l,i,i'}_{3,3}(t)V^k(X^{(0)}_t)\partial_{i'}V^m(X^{(0)}_t)' \\
&+2\sum_{l'=1}^{d}\eta^{i,k,i'}_{3,3}(t)V^l(X^{(0)}_t)\partial_{i'}V^m(X^{(0)}_t)' \\
&+\sum_{i'=1}^{d}\eta^{i',k,l,m}_{5,4}(t)\partial_{i'}V^i_0(X^{(0)}_t) + \sum_{k'=1}^{d}\eta^{i,k',l,m}_{5,4}(t)\partial_{k'}V^k_0(X^{(0)}_t) + \sum_{l'=1}^{d}\eta^{i,k,l',m}_{5,4}(t)\partial_{l'}V^l_0(X^{(0)}_t) \\
&+\sum_{m'=1}^{d}\eta^{i,k,l,m'}_{5,4}(t)\partial_{m'}V^m_0(X^{(0)}_t) + \sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta^{i,k,l,i',k'}_{5,5}(t)\partial_{i'}\partial_{k'}V^m_0(X^{(0)}_t) \\
\frac{d}{dt}\eta^{i,k,l,m,n}_{5,5}(t) &= (i\xi)\Big\{\eta^{k,l,m,n}_{4,4}(t)\hat{V}(X^{(0)}_t,t)V^i(X^{(0)}_t)' + \eta^{i,l,m,n}_{4,4}(t)\hat{V}(X^{(0)}_t,t)V^k(X^{(0)}_t)' \\
&+\eta^{i,k,m,n}_{4,4}(t)\hat{V}(X^{(0)}_t,t)V^l(X^{(0)}_t)' + \eta^{i,k,l,n}_{4,4}(t)\hat{V}(X^{(0)}_t,t)V^m(X^{(0)}_t)' \\
&+\eta^{i,k,l,m}_{4,4}(t)\hat{V}(X^{(0)}_t,t)V^n(X^{(0)}_t)'\Big\} \\
&+\eta^{l,m,n}_{3,3}(t)V^i(X^{(0)}_t)V^k(X^{(0)}_t)' + \eta^{k,m,n}_{3,3}(t)V^i(X^{(0)}_t)V^l(X^{(0)}_t)' \\
&+\eta^{k,l,n}_{3,3}(t)V^i(X^{(0)}_t)V^m(X^{(0)}_t)' + \eta^{k,l,m}_{3,3}(t)V^i(X^{(0)}_t)V^n(X^{(0)}_t)' \\
&+\eta^{i,m,n}_{3,3}(t)V^k(X^{(0)}_t)V^l(X^{(0)}_t)' + \eta^{i,l,n}_{3,3}(t)V^k(X^{(0)}_t)V^m(X^{(0)}_t)' \\
&+\eta^{i,l,m}_{3,3}(t)V^k(X^{(0)}_t)V^n(X^{(0)}_t)' + \eta^{i,k,n}_{3,3}(t)V^l(X^{(0)}_t)V^m(X^{(0)}_t)' \\
&+\eta^{i,k,m}_{3,3}(t)V^l(X^{(0)}_t)V^n(X^{(0)}_t)' + \eta^{i,k,l}_{3,3}(t)V^m(X^{(0)}_t)V^n(X^{(0)}_t)' \\
&+\sum_{i'=1}^{d}\eta^{i',k,l,m,n}_{5,5}(t)\partial_{i'}V^i_0(X^{(0)}_t) + \sum_{k'=1}^{d}\eta^{i,k',l,m,n}_{5,5}(t)\partial_{k'}V^i_0(X^{(0)}_t) \\
&+\sum_{l'=1}^{d}\eta^{i,k,l',m,n}_{5,5}(t)\partial_{l'}V^i_0(X^{(0)}_t) + \sum_{m'=1}^{d}\eta^{i,k,l,m',n}_{5,5}(t)\partial_{m'}V^i_0(X^{(0)}_t) \\
&+\sum_{n'=1}^{d}\eta^{i,k,l,m,n'}_{5,5}(t)\partial_{n'}V^i_0(X^{(0)}_t)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\eta_{6,3}^{i,k,l}(t) \;=\;& (i\xi)\Big\{2\sum_{i'=1}^{d}\eta_{5,3,2}^{i',k,l}(t)\hat{V}(X_t^{(0)},t)\partial_{i'}V^i(X_t^{(0)})' + 2\sum_{k'=1}^{d}\eta_{5,3,2}^{k',i,l}(t)\hat{V}(X_t^{(0)},t)\partial_{k'}V^k(X_t^{(0)})' \\
&+2\sum_{l'=1}^{d}\eta_{5,3,2}^{l',i,k}(t)\hat{V}(X_t^{(0)},t)\partial_{l'}V^l(X_t^{(0)})'\Big\} \\
&+4\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{4,3}^{i',k',l}(t)\partial_{i'}V^i(X_t^{(0)})\partial_{k'}V^k(X_t^{(0)})' + 4\sum_{i'=1}^{d}\sum_{l'=1}^{d}\eta_{4,3}^{i',l',k}(t)\partial_{i'}V^i(X_t^{(0)})\partial_{l'}V^l(X_t^{(0)})' \\
&+4\sum_{k'=1}^{d}\sum_{l'=1}^{d}\eta_{4,3}^{k',l',i}(t)\partial_{k'}V^k(X_t^{(0)})\partial_{l'}V^l(X_t^{(0)})' \\
&+\sum_{i'=1}^{d}\eta_{6,3}^{i',k,l}(t)\partial_{i'}V_0^i(X_t^{(0)}) + \sum_{k'=1}^{d}\eta_{6,3}^{i,k',l}(t)\partial_{k'}V_0^k(X_t^{(0)}) + \sum_{l'=1}^{d}\eta_{6,3}^{i,k,l'}(t)\partial_{l'}V_0^l(X_t^{(0)}) \\
&+\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{6,4}^{i',k',k,l}(t)\partial_{i'}\partial_{k'}V_0^i(X_t^{(0)}) + \sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{6,4}^{i',k',i,l}(t)\partial_{i'}\partial_{k'}V_0^k(X_t^{(0)}) \\
&+\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{6,4}^{i',k',i,k}(t)\partial_{i'}\partial_{k'}V_0^l(X_t^{(0)}) \\
\frac{d}{dt}\eta_{6,4}^{i,k,l,m}(t) \;=\;& (i\xi)\Big\{\eta_{5,3,2}^{k,l,m}(t)\hat{V}(X_t^{(0)},t)V^i(X_t^{(0)})' + \eta_{5,3,2}^{i,l,m}(t)\hat{V}(X_t^{(0)},t)V^k(X_t^{(0)})' \\
&+2\sum_{l'=1}^{d}\eta_{5,4}^{i,k,l',m}(t)\hat{V}(X_t^{(0)},t)\partial_{l'}V^l(X_t^{(0)})' + 2\sum_{m'=1}^{d}\eta_{5,4}^{i,k,m',l}(t)\hat{V}(X_t^{(0)},t)\partial_{m'}V^m(X_t^{(0)})'\Big\} \\
&+\eta_{4,2,2}^{l,m}(t)V^i(X_t^{(0)})V^k(X_t^{(0)})' \\
&+2\sum_{l'=1}^{d}\eta_{4,3}^{k,l',m}(t)V^i(X_t^{(0)})\partial_{l'}V^l(X_t^{(0)})' + 2\sum_{l'=1}^{d}\eta_{4,3}^{i,l',m}(t)V^k(X_t^{(0)})\partial_{l'}V^l(X_t^{(0)})' \\
&+2\sum_{m'=1}^{d}\eta_{4,3}^{k,m',l}(t)V^i(X_t^{(0)})\partial_{m'}V^m(X_t^{(0)})' + 2\sum_{m'=1}^{d}\eta_{4,3}^{i,m',l}(t)V^k(X_t^{(0)})\partial_{m'}V^m(X_t^{(0)})' \\
&+4\sum_{l'=1}^{d}\sum_{m'=1}^{d}\eta_{4,4}^{i,k,l',m'}(t)\partial_{l'}V^l(X_t^{(0)})\partial_{m'}V^m(X_t^{(0)})' \\
&+\sum_{i'=1}^{d}\eta_{6,4}^{i',k,l,m}(t)\partial_{i'}V_0^i(X_t^{(0)}) + \sum_{k'=1}^{d}\eta_{6,4}^{i,k',l,m}(t)\partial_{k'}V_0^k(X_t^{(0)}) \\
&+\sum_{l'=1}^{d}\eta_{6,4}^{i,k,l',m}(t)\partial_{l'}V_0^l(X_t^{(0)}) + \sum_{m'=1}^{d}\eta_{6,4}^{i,k,l,m'}(t)\partial_{m'}V_0^m(X_t^{(0)}) \\
&+\sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{6,5}^{i,k,i',k',m}(t)\partial_{i'}\partial_{k'}V_0^l(X_t^{(0)}) + \sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{6,5}^{i,k,l',k',l}(t)\partial_{i'}\partial_{k'}V_0^m(X_t^{(0)})
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\eta_{6,5}^{i,k,l,m,n}(t) &= (i\xi)\Big\{\eta_{5,4}^{k,l,m,n}(t)\hat{V}(X_t^{(0)},t)V^i(X_t^{(0)})' + \eta_{5,4}^{i,l,m,n}(t)\hat{V}(X_t^{(0)},t)V^k(X_t^{(0)})' \\
&+\eta_{5,4}^{i,k,m,n}(t)\hat{V}(X_t^{(0)},t)V^l(X_t^{(0)})' + \eta_{5,4}^{i,k,l,n}(t)\hat{V}(X_t^{(0)},t)V^m(X_t^{(0)})' \\
&+2\sum_{n'=1}^{d}\eta_{5,5}^{i,k,l,m,n'}(t)\hat{V}(X_t^{(0)},t)\partial_{n'}V^n(X_t^{(0)})'\Big\} \\
&+\eta_{4,3}^{l,m,n}(t)V^i(X_t^{(0)})V^k(X_t^{(0)})' + \eta_{4,3}^{k,m,n}(t)V^i(X_t^{(0)})V^l(X_t^{(0)})' + \eta_{4,3}^{k,l,n}(t)V^i(X_t^{(0)})V^m(X_t^{(0)})' \\
&+\eta_{4,3}^{i,m,n}(t)V^k(X_t^{(0)})V^l(X_t^{(0)})' + \eta_{4,3}^{i,l,n}(t)V^k(X_t^{(0)})V^m(X_t^{(0)})' + \eta_{4,3}^{i,k,n}(t)V^l(X_t^{(0)})V^m(X_t^{(0)})' \\
&+2\sum_{n'=1}^{d}\eta_{4,4}^{k,l,m,n'}(t)V^i(X_t^{(0)})\partial_{n'}V^n(X_t^{(0)})' + 2\sum_{n'=1}^{d}\eta_{4,4}^{i,l,m,n'}(t)V^k(X_t^{(0)})\partial_{n'}V^n(X_t^{(0)})' \\
&+2\sum_{n'=1}^{d}\eta_{4,4}^{i,k,m,n'}(t)V^l(X_t^{(0)})\partial_{n'}V^n(X_t^{(0)})' + 2\sum_{n'=1}^{d}\eta_{4,4}^{i,k,l,n'}(t)V^m(X_t^{(0)})\partial_{n'}V^n(X_t^{(0)})' \\
&+\sum_{i'=1}^{d}\eta_{6,5}^{i',k,l,m,n}(t)\partial_{i'}V_0^i(X_t^{(0)}) + \sum_{k'=1}^{d}\eta_{6,5}^{i,k',l,m,n}(t)\partial_{k'}V_0^k(X_t^{(0)}) \\
&+\sum_{l'=1}^{d}\eta_{6,5}^{i,k,l',m,n}(t)\partial_{l'}V_0^l(X_t^{(0)}) + \sum_{m'=1}^{d}\eta_{6,5}^{i,k,l,m',n}(t)\partial_{m'}V_0^m(X_t^{(0)}) \\
&+\sum_{n'=1}^{d}\eta_{6,5}^{i,k,l,m,n'}(t)\partial_{n'}V_0^n(X_t^{(0)}) + \sum_{i'=1}^{d}\sum_{k'=1}^{d}\eta_{6,6}^{i,k,l,m,i',k'}(t)\partial_{i'}\partial_{k'}V_0^n(X_t^{(0)}) \\
\frac{d}{dt}\eta_{6,6}^{i,k,l,m,n,o}(t) &= (i\xi)\Big\{\eta_{5,5}^{k,l,m,n,o}(t)\hat{V}(X_t^{(0)},t)V^i(X_t^{(0)})' + \eta_{5,5}^{i,l,m,n,o}(t)\hat{V}(X_t^{(0)},t)V^k(X_t^{(0)})' \\
&+\eta_{5,5}^{i,k,m,n,o}(t)\hat{V}(X_t^{(0)},t)V^l(X_t^{(0)})' + \eta_{5,5}^{i,k,l,n,o}(t)\hat{V}(X_t^{(0)},t)V^m(X_t^{(0)})' \\
&+\eta_{5,5}^{i,k,m,l,o}(t)\hat{V}(X_t^{(0)},t)V^n(X_t^{(0)})' + \eta_{5,5}^{i,k,m,l,n}(t)\hat{V}(X_t^{(0)},t)V^o(X_t^{(0)})'\Big\} \\
&+\eta_{4,4}^{l,m,n,o}(t)V^i(X_t^{(0)})V^k(X_t^{(0)})' + \eta_{4,4}^{k,m,n,o}(t)V^i(X_t^{(0)})V^l(X_t^{(0)})' \\
&+\eta_{4,4}^{k,l,n,o}(t)V^i(X_t^{(0)})V^m(X_t^{(0)})' + \eta_{4,4}^{k,l,m,o}(t)V^i(X_t^{(0)})V^n(X_t^{(0)})' \\
&+\eta_{4,4}^{k,l,m,n}(t)V^i(X_t^{(0)})V^o(X_t^{(0)})' + \eta_{4,4}^{i,m,n,o}(t)V^k(X_t^{(0)})V^l(X_t^{(0)})' \\
&+\eta_{4,4}^{i,l,n,o}(t)V^k(X_t^{(0)})V^m(X_t^{(0)})' + \eta_{4,4}^{i,l,m,o}(t)V^k(X_t^{(0)})V^n(X_t^{(0)})' \\
&+\eta_{4,4}^{i,l,m,n}(t)V^k(X_t^{(0)})V^o(X_t^{(0)})' + \eta_{4,4}^{i,k,n,o}(t)V^l(X_t^{(0)})V^m(X_t^{(0)})' \\
&+\eta_{4,4}^{i,k,m,o}(t)V^l(X_t^{(0)})V^n(X_t^{(0)})' + \eta_{4,4}^{i,k,m,n}(t)V^l(X_t^{(0)})V^o(X_t^{(0)})' \\
&+\eta_{4,4}^{i,k,l,o}(t)V^m(X_t^{(0)})V^n(X_t^{(0)})' + \eta_{4,4}^{i,k,l,n}(t)V^m(X_t^{(0)})V^o(X_t^{(0)})' \\
&+\eta_{4,4}^{i,k,l,m}(t)V^n(X_t^{(0)})V^o(X_t^{(0)})' \\
&+\sum_{i'=1}^{d}\eta_{6,6}^{i',k,l,m,n,o}(t)\partial_{i'}V_0^i(X_t^{(0)}) + \sum_{k'=1}^{d}\eta_{6,6}^{i,k',l,m,n,o}(t)\partial_{k'}V_0^k(X_t^{(0)}) \\
&+\sum_{l'=1}^{d}\eta_{6,6}^{i,k,l',m,n,o}(t)\partial_{l'}V_0^l(X_t^{(0)}) + \sum_{m'=1}^{d}\eta_{6,6}^{i,k,l,m',n,o}(t)\partial_{m'}V_0^m(X_t^{(0)}) \\
&+\sum_{n'=1}^{d}\eta_{6,6}^{i,k,l,m,n',o}(t)\partial_{n'}V_0^n(X_t^{(0)}) + \sum_{o'=1}^{d}\eta_{6,6}^{i,k,l,m,n,o'}(t)\partial_{o'}V_0^o(X_t^{(0)})
\end{aligned}
$$

From the derivation of differential equations, it is easily shown that each $\eta_{j,m,k}(T)$ is expressed as a polynomial of degree j with respect to $(i\xi)$, and also it is shown that $\mathbf{E}[X^{j,m,k}Z_T^{\langle\xi\rangle}]$ is a polynomial of $(i\xi)$ of degree less than or equal to $2j$.

We summarize the discussion above as the following theorem:

Theorem 2 *The asymptotic expansion of density of $G^{(\epsilon)}$ up to ϵ^3-order is given by*

$$
\begin{aligned}
f_{G^{(\epsilon)}}(x) &= f_{g_{1T}}(x) \\
&\quad +\epsilon \left\{ \sum_{l=1}^{3} C_{1l} H_l(x;\Sigma_T) \right\} f_{g_{1T}}(x) \\
&\quad +\epsilon^2 \left\{ \sum_{l=1}^{6} C_{2l} H_l(x;\Sigma_T) \right\} f_{g_{1T}}(x) \\
&\quad +\epsilon^3 \left\{ \sum_{l=1}^{9} C_{3l} H_l(x;\Sigma_T) \right\} f_{g_{1T}}(x) \\
&\quad +o(\epsilon^3).
\end{aligned}
$$

where

$$
\begin{aligned}
& C_{1l} = \Sigma_T a_{l-1}^{1,1,(1)}, \\
& C_{21} = \Sigma_T a_0^{2,1,(0,1)}, \quad C_{2l} = \Sigma_T a_{l-1}^{2,1,(0,1)} + \frac{1}{2}\Sigma_T^2 a_{l-2}^{2,2,(2,0)} (l \geq 2), \\
& C_{31} = \Sigma_T a_0^{3,1,(0,0,1)}, \quad C_{32} = \Sigma_T a_1^{3,1,(0,0,1)} + \frac{1}{2}\Sigma_T^2 a_0^{3,2,(1,2,0)}, \\
& C_{3l} = \Sigma_T a_{l-1}^{3,1,(0,0,1)} + \frac{1}{2}\Sigma_T^2 a_{l-2}^{3,2,(1,2,0)} + \frac{1}{6}\Sigma_T^3 a_{l-2}^{3,3,(3,0,0)} (l \geq 3).
\end{aligned}
$$

$a_l^{j,m,k}$ *are given by (29), and expectations in (29) are obtained as the solutions to the system of ordinary differential equations given in Proposition 2.*

[General expressions for the system of ODEs and the asymptotic expansion of the density of $G^{(\epsilon)}$]

In the following, we will extend Propsition 2 as well as Theorem 2: Specifically, we will derive general expressions for the system of ODEs in the proposition and the asymptotic expansion of the density of $G^{(\epsilon)}$ in the theorem.

First, recall that $X^{(\epsilon)} = (X^{(\epsilon),1}, \cdots, X^{(\epsilon),d})$ is given by

$$
dX_t^{(\epsilon),j} = V_0^j(X_t^{(\epsilon)}, \epsilon)dt + \epsilon V^j(X_t^{(\epsilon)})dW_t.
$$

Hereafter, redefine A_{kt} as $A_{kt} = \frac{1}{k!}\frac{\partial^k X_t^{(\epsilon)}}{\partial \epsilon^k}|_{\epsilon=0}$ as in Section 2, which is slightly different from the defintion so far in this section, $A_{kt} = \frac{\partial^k X_t^{(\epsilon)}}{\partial \epsilon^k}|_{\epsilon=0}$.

Also, A_{kt}^j, $j = 1, \cdots, d$ denote the j-th elements of A_{kt}. In particular, A_{1t} is represented by

$$
A_{1t} = \int_0^t Y_t Y_u^{-1}\left(\partial_\epsilon V_0(X_u^{(0)}, 0)du + V(X_u^{(0)})dW_u\right),
$$

where Y denotes the solution to the differential equation;

$$
dY_t = \partial V_0(X_t^{(0)}, 0)Y_t dt;\ Y_0 = I_d.
$$

Here, ∂V_0 denotes the $d \times d$ matrix whose (j,k)-element is $\partial_k V_0^j = \frac{\partial V_0^j(x,\epsilon)}{\partial x_k}$, V_0^j is the j-th element of V_0, and I_d denotes the $d \times d$ identity matrix.

For $k \geq 2$, A^j_{kt}, $j = 1, \cdots, d$ is recursively determined by the following:

$$
\begin{aligned}
A^j_{kt} &= \frac{1}{k!}\int_0^t \partial^k_\epsilon V^j_0(X^{(0)}, 0)du \\
&+ \sum_{l=1}^{k} \sum_{\vec{l}_\beta, \vec{d}_\beta}^{(l)} \frac{1}{(k-l)!}\frac{1}{\beta!}\int_0^t \left(\prod_{j=1}^{\beta} A^{d_j}_{l_j u}\right) \partial^\beta_{\vec{d}_\beta} \partial^{k-l}_\epsilon V^j_0(X^{(0)}_u, 0)du \\
&+ \sum_{\vec{l}_\beta, \vec{d}_\beta}^{(k-1)} \frac{1}{\beta!}\int_0^t \left(\prod_{j=1}^{\beta} A^{d_j}_{l_j u}\right) \partial^\beta_{\vec{d}_\beta} V^j(X^{(0)}_u)dW_u,
\end{aligned}
$$

where $\partial^l_\epsilon = \frac{\partial^l}{\partial \epsilon^l}$, $\partial^\beta_{\vec{d}_\beta} = \frac{\partial^\beta}{\partial x_{d_1} \cdots \partial x_{d_\beta}}$,

$$
L_{n,\beta} = \left\{ \vec{l}_\beta = (l_1, \cdots, l_\beta); \sum_{j=1}^{\beta} l_j = n, l_j \geq 1, j = 1, \cdots, \beta \right\}
$$

and

$$
\sum_{\vec{l}_\beta, \vec{d}_\beta}^{(n)} = \sum_{\beta=1}^{n} \sum_{\vec{l}_\beta \in L_{n,\beta}} \sum_{\vec{d}_\beta \in \{1,\cdots,d\}^\beta}
$$

for $n \geq 1$, and

$$
\sum_{\vec{l}_\beta, \vec{d}_\beta}^{(0)} = \sum_{\beta=0} \sum_{\vec{l}_0=(\emptyset)} \sum_{\vec{d}_0=(\emptyset)}.
$$

Then, g_{0T} and g_{1T} can be written as

$$
\begin{aligned}
g_{0T} &= g(X^{(0)}_T), \\
g_{1T} &= \sum_{j=1}^{d} \partial_j g(X^{(0)}_T) A^j_{1T}.
\end{aligned}
$$

For $n \geq 2$, g_{nT} is expressed as follows:

$$
g_{nT} = \sum_{\vec{l}_\beta \vec{d}_\beta}^{(n)} \frac{1}{\beta!} \partial^\beta_{\vec{d}_\beta} g(X^{(0)}_T) A^{d_1}_{l_1 T} \cdots A^{d_\beta}_{l_\beta T}.
$$

Then, for $G^{(\epsilon)} = \frac{g(X^{(\epsilon)}_T) - g_{0T}}{\epsilon} (\epsilon \in (0, 1])$, we have the following expression of $\mathbf{E}[\Phi(G^{(\epsilon)})]$, $\Phi \in \mathcal{S}'$ under Assumption 1:

$$
\begin{aligned}
\mathbf{E}[\Phi(G^{(\epsilon)})] &= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!} \int_{\mathbf{R}} \Phi(x)(-1)^\delta \frac{d^\delta}{dx^\delta} \left\{ \mathbf{E}[X^{\vec{k}_\delta} | g_{1T} = x] f_{g_{1T}}(x) \right\} dx + o(\epsilon^N) \\
&= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!} \int_{\mathbf{R}} \Phi(x)(-1)^\delta \frac{d^\delta}{dx^\delta} \left\{ \sum_{l=0}^{\infty} \frac{a^{\vec{k}_\delta}_l}{\Sigma^l_T} H_l(x - C, \Sigma_T) f_{g_{1T}}(x) \right\} dx + o(\epsilon^N) \\
&= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!} \int_{\mathbf{R}} \Phi(x) \left\{ \sum_{l=0}^{\infty} \frac{a^{\vec{k}_\delta}_l}{\Sigma^{l+\delta}_T} H_{l+\delta}(x - C, \Sigma_T) f_{g_{1T}}(x) \right\} dx + o(\epsilon^N)
\end{aligned}
$$

where

$$f_{g_{1T}}(x) = \frac{1}{\sqrt{2\pi\Sigma_T}}\exp\left(-\frac{(x-C)^2}{2\Sigma_T}\right),$$

$$C := \left(\partial g(X_T^{(0)})\right)' \int_0^T Y_T Y_t^{-1}\partial_\epsilon V_0(X_t^{(0)}, 0)dt,$$

$$X^{\vec{k}_\delta} = \prod_{j=1}^{\delta} g_{(k_j+1)T};\ (\vec{k}_\delta \in L_{n,\delta}),$$

$$\sum_{\vec{k}_\delta}^{(n)} = \sum_{\delta=1}^{n}\sum_{\vec{k}_\delta \in L_{n,\delta}},$$

and

$$a_l^{\vec{k}_\delta} = \frac{1}{l!}\frac{1}{i^l}\frac{\partial^l}{\partial \xi^l}\mathbf{E}[X^{\vec{k}_\delta}Z_T^{\langle\xi\rangle}]\bigg|_{\xi=0}.$$

Here, recall that we have defined $Z^{\langle\xi\rangle} = \{Z_t^{\langle\xi\rangle}; t \in \mathbf{R}^+\}$ as the stochastic process:

$$Z_t^{\langle\xi\rangle} = \exp\{i\xi\hat{g}_{1t} + \frac{\xi^2}{2}\Sigma_t\},$$

where

$$\hat{g}_{1t} = \int_0^t \hat{V}(X_u^{(0)}, u)dW_u.$$

In particular, let Φ be the delta function at $x \in \mathbf{R}$, δ_x, we obtain the asymptotic expansion of density of $G^{(\epsilon)}$:

$$\begin{aligned} f_{G^{(\epsilon)}}(x) &= \mathbf{E}[\delta_x(G^{(\epsilon)})] \\ &= \sum_{n=0}^{N}\epsilon^n \sum_{\vec{k}_\delta}^{(n)}\frac{1}{\delta!}\sum_{l=0}^{\infty}\frac{a_l^{\vec{k}_\delta}}{\Sigma_T^{l+\delta}}H_{l+\delta}(x-C,\Sigma_T)f_{g_{1T}}(x) + o(\epsilon) \quad (32)\end{aligned}$$

Next, define $\eta_{\vec{l}_\beta}^{\vec{d}_\beta}(t;\xi)$ for $\vec{l}_\beta \in L_{n,\beta}$ and $\vec{d}_\beta \in \{1,\cdots,d\}^\beta$ $(n \geq \beta \geq 1)$ as

$$\eta_{\vec{l}_\beta}^{\vec{d}_\beta}(t;\xi) = \mathbf{E}\left[\left(\prod_{j=1}^{\beta}A_{l_j t}^{d_j}\right)Z_t^{\langle\xi\rangle}\right], \quad (33)$$

and for $n=0$ as

$$\eta_{(\emptyset)}^{(\emptyset)}(t;\xi) = \mathbf{E}\left[Z_t^{\langle\xi\rangle}\right]. \quad (34)$$

Then, unconditional expectations $\mathbf{E}[X^{\vec{k}_\delta}Z_T^{\langle\xi\rangle}]$ appeared in $a_l^{\vec{k}_\delta}$ can be written in terms of η as follows:

$$\begin{aligned} \mathbf{E}[X^{\vec{k}_\delta}Z_T^{\langle\xi\rangle}] &= \mathbf{E}\left[\left(\prod_{j=1}^{\delta}g_{(k_j+1)T}\right)Z_T^{\langle\xi\rangle}\right] \\ &= \mathbf{E}\left[\left(\prod_{j=1}^{\delta}\sum_{\vec{l}_{\beta_j}^j,\vec{d}_{\beta_j}^j}^{(k_j+1)}\frac{1}{\beta_j!}\partial_{\vec{d}_{\beta_j}^j}^{\beta_j}g(X_T^{(0)})A_{l_1^j T}^{d_1^j}\cdots A_{l_{\beta_j}^j}^{d_{\beta_j}^j}\right)Z_T^{\langle\xi\rangle}\right] \\ &= \sum_{\vec{l}_{\beta_1}^1,\vec{d}_{\beta_1}^1}^{(k_1+1)}\cdots\sum_{\vec{l}_{\beta_\delta}^\delta,\vec{d}_{\beta_\delta}^\delta}^{(k_\delta+1)}\left(\prod_{j=1}^{\delta}\frac{1}{\beta_j!}\partial_{\vec{d}_{\beta_j}^j}^{\beta_j}g(X_T^{(0)})\right)\eta_{\vec{l}_{\beta_1}^1\otimes\cdots\otimes\vec{l}_{\beta_\delta}^\delta}^{\vec{d}_{\beta_1}^1\otimes\cdots\otimes\vec{d}_{\beta_\delta}^\delta}(T;\xi) \end{aligned}$$

(35)

where

$$\vec{d}^i_{\beta_i} \otimes \vec{d}^j_{\beta_j} := (d^i_1, \cdots, d^i_{\beta_i}, d^j_1, \cdots, d^j_{\beta_j}),$$
$$\vec{l}^i_{\beta_i} \otimes \vec{l}^j_{\beta_j} := (l^i_1, \cdots, l^i_{\beta_i}, l^j_1, \cdots, l^j_{\beta_j}).$$

So, we have to calculate $\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(T;\xi)$ to evaluate the asymptotic expansion formula (32).

The following theorem provides a way to calculate general $\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(T;\xi)$ as a solution to the system of ordinary differential equations:

Theorem 3 *For $\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(t;\xi)$ defined in (33), the following system of ordinary differential equations is hold:*

$$\begin{aligned}
\frac{d}{dt}\left\{\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(t;\xi)\right\} &= \sum_{k=1}^{\beta}\frac{1}{l_k!}\left\{\eta^{\vec{d}_{\beta/k}}_{\vec{l}_{\beta/k}}(t;\xi)\right\}\left\{\partial^{l_k}_\epsilon V^{d_k}_0(X^{(0)}_t)\right\} \\
&+ \sum_{k=1}^{\beta}\sum_{l=1}^{l_k}\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l)}\frac{1}{(l_k-l)!}\frac{1}{\gamma!}\left\{\eta^{(\vec{d}_{\beta/k})\otimes\vec{\tilde{d}}_\gamma}_{(\vec{l}_{\beta/k})\otimes\vec{m}_\gamma}(t;\xi)\right\}\left\{\partial^\gamma_{\vec{\tilde{d}}_\gamma}\partial^{l_k-l}_\epsilon V^{d_k}_0(X^{(0)}_t)\right\} \\
&+ \sum_{\substack{k,m=1\\k<m}}^{\beta}\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l_k-1)}\sum_{\vec{m}_\delta,\vec{\tilde{d}}_\delta}^{(l_m-1)}\frac{1}{\gamma!\delta!}\left\{\eta^{(\vec{d}_{\beta/k,m})\otimes\vec{\tilde{d}}_\gamma\otimes\vec{\tilde{d}}_\delta}_{(\vec{l}_{\beta/k,m})\otimes\vec{m}_\gamma\otimes\vec{m}_\delta}(t;\xi)\right\}\left\{\partial^\gamma_{\vec{\tilde{d}}_\gamma}V^{d_k}(X^{(0)}_t)\right\}\left\{\partial^\delta_{\vec{\tilde{d}}_\delta}V^{d_m}(X^{(0)}_t)\right\} \\
&+ (i\xi)\sum_{k=1}^{\beta}\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l_k-1)}\frac{1}{\gamma!}\left\{\eta^{(\vec{d}_{\beta/k})\otimes\vec{\tilde{d}}_\gamma}_{(\vec{l}_{\beta/k})\otimes\vec{m}_\gamma}(t;\xi)\right\}\left\{\partial^\gamma_{\vec{\tilde{d}}_\gamma}V^{d_k}(X^{(0)}_t)\right\}\hat{V}(X^{(0)}_t,t), \qquad (36)
\end{aligned}$$

where

$$\vec{l}_{\beta/k} := (l_1,\cdots,l_{k-1},l_{k+1},\cdots,l_\beta)$$
$$\vec{l}_{\beta/k,n} := (l_1,\cdots,l_{k-1},l_{k+1},\cdots,l_{n-1},l_{n+1},\cdots,l_\beta),\ 1\le k<n\le\beta$$
$$\vec{l}_\beta\otimes\vec{m}_\gamma := (l_1,\cdots,l_\beta,m_1,\cdots,m_\gamma)$$

for $\vec{l}_\beta = (l_1,\cdots,l_\beta)$ and $\vec{m}_\gamma = (m_1,\cdots,m_\gamma)$.

(Proof) First, Applying Itô's formula to $\left(\prod_{j=1}^{\beta} A_{l_j t}^{d_j}\right)$, we have

$$\begin{aligned}
d\left(\prod_{j=1}^{\beta} A_{l_j t}^{d_j}\right) &= \sum_{k=1}^{\beta}\left(\prod_{\substack{j=1\\ j\neq k}}^{\beta} A_{l_j t}^{d_j}\right) dA_{l_k t}^{d_k} + \sum_{\substack{k,m=1\\ k<m}}^{\beta}\left(\prod_{\substack{j=1\\ j\neq k,m}}^{\beta} A_{l_j t}^{d_j}\right) d\langle A_{l_k}^{d_k}, A_{l_m}^{d_m}\rangle_t \\
&= \sum_{k=1}^{\beta}\left(\prod_{\substack{j=1\\ j\neq k}}^{\beta} A_{l_j t}^{d_j}\right) \frac{1}{l_k!}\partial_{\epsilon}^{l_k} V_0^{d_k}(X_t^{(0)})dt \\
&+ \sum_{k=1}^{\beta}\sum_{l=1}^{l_k}\left(\prod_{\substack{j=1\\ j\neq k}}^{\beta} A_{l_j t}^{d_j}\right)\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l)} \frac{1}{(l_k-l)!}\frac{1}{\gamma!}\left(\prod_{j'=1}^{\gamma} A_{m_{j'} t}^{\tilde{d}_{j'}}\right)\partial_{\tilde{d}_\gamma}^{\gamma}\partial_{\epsilon}^{l_k-l} V_0^{d_k}(X_t^{(0)})dt \\
&+ \sum_{k=1}^{\beta}\left(\prod_{\substack{j=1\\ j\neq k}}^{\beta} A_{l_j t}^{d_j}\right)\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l_k-1)} \frac{1}{\gamma!}\left(\prod_{j'=1}^{\gamma} A_{m_{j'} t}^{\tilde{d}_{j'}}\right)\partial_{\tilde{d}_\gamma}^{\gamma} V^{d_k}(X_t^{(0)})dW_t \\
&+ \sum_{\substack{k,m=1\\ k<m}}^{\beta}\left(\prod_{\substack{j=1\\ j\neq k,m}}^{\beta} A_{l_j t}^{d_j}\right)\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l_k-1)}\sum_{\vec{m}_\delta,\vec{\hat{d}}_\delta}^{(l_m-1)} \frac{1}{\gamma!\delta!} \\
&\quad\times\left(\prod_{j'=1}^{\gamma} A_{m_{j'} t}^{\tilde{d}_{j'}}\right)\partial_{\tilde{d}_\gamma}^{\gamma} V^{d_k}(X_t^{(0)})\left(\prod_{j'=1}^{\delta} A_{m_{j'} t}^{\hat{d}_{j'}}\right)\partial_{\hat{d}_\delta}^{\delta} V^{d_m}(X_t^{(0)})dt.
\end{aligned} \tag{37}$$

Note also that

$$dZ_t^{\langle\xi\rangle} = (i\xi)\hat{V}(X_t^{(0)}, t)Z_t^{\langle\xi\rangle}dW_t. \tag{38}$$

Then, applying Itô's formula again to $\left(\prod_{j=1}^{\beta} A_{l_j t}^{d_j} Z_t^{\langle\xi\rangle}\right)$ and taking expectations on both sides, we obtain the result. □

Remark 1 *Note that due to the structure of the system of equations and* $\eta_{(\emptyset)}^{(\emptyset)}(t;\xi) = \mathbf{E}[Z_t^{\langle\xi\rangle}] = 1$ *it is easily shown by induction that each* $\eta_{\vec{l}_\beta}^{\vec{d}_\beta}(t;\xi)$ *is expressed as a polynomial of degree* $n = \sum_{j=1}^{\beta} l_j$ *with respect to* $(i\xi)$.

From Remark 1, $\mathbf{E}[X^{\vec{k}_\delta} Z_T^{\langle\xi\rangle}]$ is a polynomial of degree $(n+\delta)$ with respect to $(i\xi)$ for $\vec{k}_\delta \in L_{n,\delta}$. So, we can see that $a_l^{\vec{k}_\delta} = 0 (l > n+\delta)$, and thus, the most inner infinite sum in the expression (32) is actually a finite sum.

Finally, the following theorem presents a general expression for the asymptotic expansion of the density function of $G^{(\epsilon)}$.

Theorem 4 *The asymptotic expansion of the density function of* $G^{(\epsilon)}$ *up to* ϵ^N*-order is given by*

$$f_{G^{(\epsilon)}}(x) = f_{g_{1T}}(x) + \sum_{n=1}^{N}\epsilon^n\left(\sum_{m=0}^{3n} C_{nm}H_m(x-C,\Sigma_T)\right)f_{g_{1T}}(x) + o(\epsilon^N) \tag{39}$$

where

$$
\begin{aligned}
C_{nm} &= \frac{1}{\Sigma_T^m}\sum_{\delta=1}^{m}\sum_{\vec{k}_\delta\in L_{n,\delta}}\sum_{\vec{l}^1_{\beta_1},\vec{d}^1_{\beta_1}}^{(k_1+1)}\cdots\sum_{\vec{l}^\delta_{\beta_\delta},\vec{d}^\delta_{\beta_\delta}}^{(k_\delta+1)}\frac{1}{\delta!(m-\delta)!} \\
&\left(\prod_{j=1}^{\delta}\frac{1}{\beta_j!}\partial^{\beta_j}_{\vec{d}^j_{\beta_j}}g(X_T^{(0)})\right)\frac{1}{i^{m-\delta}}\frac{\partial^{m-\delta}}{\partial\xi^{m-\delta}}\left\{\eta^{\vec{d}^1_{\beta_1}\otimes\cdots\otimes\vec{d}^\delta_{\beta_\delta}}_{\vec{l}^1_{\beta_1}\otimes\cdots\otimes\vec{l}^\delta_{\beta_\delta}}(T;\xi)\right\}\Bigg|_{\xi=0}
\end{aligned} \tag{40}
$$

and $\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(T;\xi)$ *are obtained as a solution to the system of ODEs given in Theorem 3.*

4.2 Asymptotic Expansion of Option Prices

We apply the asymptotic expansion to option pricing. We consider the plain vanilla option on the underlying asset $g(X_T^{(\epsilon)})$ whose dynamics is given by (1). For example, the call option price with strike K and maturity T is given by

$$
C(K,T) = \epsilon P(0,T)\int_{-k^{(\epsilon)}}^{\infty}(x+k^{(\epsilon)})f_{G^{(\epsilon)}}(x)dx
$$

where $k^{(\epsilon)} = \frac{G^{(0)}-K}{\epsilon}$, $P(0,T)$ denotes the price at time 0 of a zero coupon bond with maturity T, and $f_{G^{(\epsilon)}}$ is the normal asymptotic expansion of density of $G^{(\epsilon)}$ given by (28). In particular, using the result of the previous subsection, the approximated price to the option up to the fourth order can be expressed as

$$
\begin{aligned}
C(K,T) &= \epsilon P(0,T)\int_{-k^{(\epsilon)}}^{\infty}(x+k^{(\epsilon)})f_{g_{1T}}(x)dx \\
&+\epsilon^2 P(0,T)\int_{-k^{(\epsilon)}}^{\infty}(x+k^{(\epsilon)})\left\{\sum_{l=1}^{3}C_{1l}H_l(x;\Sigma_T)\right\}f_{g_{1T}}(x)dx \\
&+\epsilon^3 P(0,T)\int_{-k^{(\epsilon)}}^{\infty}(x+k^{(\epsilon)})\left\{\sum_{l=1}^{6}C_{2l}H_l(x;\Sigma_T)\right\}f_{g_{1T}}(x)dx \\
&+\epsilon^4 P(0,T)\int_{-k^{(\epsilon)}}^{\infty}(x+k^{(\epsilon)})\left\{\sum_{l=1}^{9}C_{3l}H_l(x;\Sigma_T)\right\}f_{g_{1T}}(x)dx \\
&+o(\epsilon^4).
\end{aligned}
$$

Integrals appeared in the right hand side can be calcurated using following formulas related to the Hermite polynomial

$$
\begin{aligned}
\int_{-y}^{\infty}H_k(x;\Sigma)f_{g_{1T}}(x)dx &= \Sigma H_{k-1}(-y;\Sigma)f_{g_{1T}}(y)\ (k\geq 1), \\
\int_{-y}^{\infty}xH_k(x;\Sigma)f_{g_{1T}}(x)dx &= -\Sigma y H_{k-1}(-y;\Sigma)f_{g_{1T}}(y) \\
&\quad +\Sigma^2 H_{k-2}(-y;\Sigma)f_{g_{1T}}(y)\ (k\geq 2).
\end{aligned}
$$

4.3 Log-Normal Asymptotic Expansion

Suppose that the underlying asset process $S^{(\epsilon)}$ follows

$$
\begin{aligned}
dS_t^{(\epsilon)} &= g(X_t^{(\epsilon)})S_t^{(\epsilon)}\bar{\sigma}dW_t; \quad S_0^{(\epsilon)} = s_0 \\
dX_t^{(\epsilon)} &= V_0(X_t^{(\epsilon)},\epsilon)dt+\epsilon V(X_t^{(\epsilon)})dW_t; \quad X_0^{(\epsilon)} = x_0\in\mathbf{R}^d.
\end{aligned}
$$

Define $\hat{X}^{(\epsilon)}$ as

$$\hat{X}_t^{(\epsilon)} = \log\left(\frac{S_t^{(\epsilon)}}{s_0}\right).$$

Then, we have

$$\hat{X}_t^{(\epsilon)} = -\frac{|\bar{\sigma}|^2}{2}\int_0^t g(X_u^{(\epsilon)})^2 du + \int_0^t g(X_u^{(\epsilon)})\bar{\sigma} dW_u,$$

and note that

$$\hat{X}_T^{(0)} \sim N(\hat{\mu}_T, \hat{\Sigma}_T),$$

where

$$\begin{aligned}\hat{\mu}_T &= -\frac{|\bar{\sigma}|^2}{2}\int_0^T g(X_u^{(0)})^2 du = -\frac{1}{2}\hat{\Sigma}_T,\\ \hat{\Sigma}_T &= |\bar{\sigma}|^2\int_0^T g(X_u^{(0)})^2 du.\end{aligned}$$

Moreover, an asymptotic expansion of $\hat{X}_T^{(\epsilon)}$ up to ϵ^N-order is expressed as

$$\hat{X}_T^{(\epsilon)} = \hat{X}_T^{(0)} + \sum_{n=1}^{N}\frac{\epsilon^n}{n!}\hat{A}_{nT} + o(\epsilon^N),$$

where $\hat{A}_{nt} = \frac{\partial^n \hat{X}_t^{(\epsilon)}}{\partial \epsilon^n}|_{\epsilon=0}$. Note that $S^{(\epsilon)}$ is expanded around a log-normal distribution since $\hat{X}_T^{(0)}$ has a Gaussian distribution.

Next, define $Z_t^{\langle\xi\rangle}$ as

$$Z_t^{\langle\xi\rangle} = \exp\left(i\xi\int_0^t g(X_u^{(0)})\bar{\sigma} dW_u\right).$$

Then, the result in the previous subsection is applied to deriving the density function of $\hat{X}_T^{(\epsilon)}$ if $G^{(\epsilon)}$ is replaced by $\hat{X}_T^{(\epsilon)}$.

Similar to the normal case, the log-normal asymptotic expansion of the price of the call option on $\hat{X}_T^{(\epsilon)}$ is given by

$$C(K,T) = P(0,T)\int_{\log\frac{K}{s_0}}^{\infty}(s_0 e^x - K) f_{\hat{X}_T^{(\epsilon)}}(x)dx$$

5 Numerical Examples

5.1 λ-SABR model

To test the validity of the expansion, we first consider the European plain-vanilla call and put prices under the following λ-SABR model [24] (interest rate=0%) :

$$\begin{aligned}dS^{(\epsilon)}(t) &= \epsilon\sigma^{(\epsilon)}(t)(S^{(\epsilon)}(t))^{\beta} dW_t^1,\\ d\sigma^{(\epsilon)}(t) &= \lambda(\theta - \sigma^{(\epsilon)}(t))dt + \epsilon\nu_1\sigma^{(\epsilon)}(t)dW_t^1 + \epsilon\nu_2\sigma^{(\epsilon)}(t)dW_t^2,\end{aligned}$$

where $\nu_1 = \rho\nu$, $\nu_2 = (\sqrt{1-\rho^2})\nu$. (The correlation between S and σ is $\rho \in [-1,1]$.)

Approximated prices by the asymptotic expansion method are calculated up to the fifth order. Note that all the solutions to differential equations are

Table 1:

Parameter	$S(0)$	λ	$\sigma(0)$	β	ρ	θ	ν	T
i	100	0.1	3.0	0.5	-0.7	3.0	0.3	10
ii	100	0.1	3.0	0.5	-0.7	3.0	0.1	10
iii	100	0.1	3.0	0.5	-0.7	3.0	0.3	1

obtained analytically. Benchmark values are computed by Monte Carlo simulations. ϵ is set to be one and other parameters used in the test are given in Table 1:

In Monte Carlo simulations for benchmark values, we use Euler-Maruyama scheme as a discretization scheme with 1024, 1024, and 512 time steps for case i, ii, and iii respectively, and generate 10^8 paths in each simulation.

For the case of $\beta = 1$ in the λ-SABR model, we can apply the log-normal asymptotic expansion method given in the previous section. To test the efficiency of the high order log-normal asymptotic expansion method, we consider the European plain-vanilla call and put prices under the following parameters (and $\epsilon = 1$ as well as in the previous examples) with different maturities:

Table 2:

Parameter	$S(0)$	λ	$\sigma(0)$	β	ρ	θ	ν	T
iv	100	0.1	0.3	1.0	-0.7	0.3	0.3	10
v	100	0.1	0.3	1.0	-0.7	0.3	0.3	20
vi	100	0.1	0.3	1.0	-0.7	0.3	0.3	30

We calculate approximated prices by the log-normal asymptotic expansion method up to the fourth order. Benchmark prices are computed by Monte Carlo simulations. In the simulations, we adapt the second order discretization scheme given by Ninomiya-Victoir [34] with 128, 256, 256 time steps respectively.

Results are in Table 3 and Table 4.

From the results, in each case, the higher order asymptotic expansion or log-normal asymptotic expansion almost always improve the accuracy of approximation by the lower expansions. Improvement is significant especially in long-term cases in which the lower order asymptotic expansions cannot approximate the price well.

Table 3: Monte Carlo prices and approximation errors by the asymptotic expansions in case i, ii, and iii.

			A.E.(Difference)					A.E.(Relative Difference)				
Case	Strike(C/P)	MC	1st	2nd	3rd	4th	5th	1st	2nd	3rd	4th	5th
i	50 Put	13.109	4.876	5.000	2.313	1.067	0.260	37.20 %	38.14 %	17.64 %	8.14 %	1.98 %
	60 Put	16.618	4.544	4.648	1.931	0.938	0.195	27.34 %	27.97 %	11.62 %	5.65 %	1.17 %
	70 Put	20.482	4.241	4.322	1.585	0.844	0.149	20.71 %	21.10 %	7.74 %	4.12 %	0.73 %
	80 Put	24.720	3.965	4.020	1.269	0.778	0.117	16.04 %	16.26 %	5.14 %	3.15 %	0.47 %
	90 Put	29.347	3.710	3.738	0.980	0.735	0.094	12.64 %	12.74 %	3.34 %	2.51 %	0.32 %
	100 Call	34.375	3.472	3.472	0.712	0.712	0.077	10.10 %	10.10 %	2.07 %	2.07 %	0.22 %
	110 Call	29.811	3.246	3.217	0.459	0.704	0.063	10.89 %	10.79 %	1.54 %	2.36 %	0.21 %
	120 Call	25.659	3.026	2.971	0.220	0.711	0.050	11.79 %	11.58 %	0.86 %	2.77 %	0.19 %
	130 Call	21.914	2.809	2.728	-0.010	0.731	0.035	12.82 %	12.45 %	-0.04 %	3.33 %	0.16 %
	140 Call	18.571	2.591	2.487	-0.230	0.762	0.018	13.95 %	13.39 %	-1.24 %	4.10 %	0.10 %
	150 Call	15.615	2.370	2.246	-0.441	0.804	-0.002	15.18 %	14.38 %	-2.83 %	5.15 %	-0.02 %
ii	50 Put	1.682	-0.914	0.030	0.475	0.182	-0.016	-54.33 %	1.81 %	28.25 %	10.84 %	-0.92 %
	60 Put	2.607	-1.056	0.129	0.445	0.103	-0.003	-40.52 %	4.94 %	17.06 %	3.95 %	-0.13 %
	70 Put	3.950	-1.047	0.214	0.364	0.061	0.008	-26.51 %	5.41 %	9.22 %	1.55 %	0.20 %
	80 Put	5.883	-0.825	0.254	0.258	0.048	0.013	-14.03 %	4.32 %	4.39 %	0.82 %	0.23 %
	90 Put	8.631	-0.390	0.237	0.150	0.047	0.016	-4.52 %	2.75 %	1.74 %	0.54 %	0.18 %
	100 Call	12.450	0.166	0.166	0.048	0.048	0.018	1.33 %	1.33 %	0.39 %	0.39 %	0.14 %
	110 Call	7.577	0.664	0.037	-0.050	0.053	0.022	8.76 %	0.49 %	-0.67 %	0.70 %	0.29 %
	120 Call	4.131	0.927	-0.153	-0.149	0.062	0.027	22.43 %	-3.70 %	-3.60 %	1.49 %	0.65 %
	130 Call	2.008	0.894	-0.367	-0.217	0.086	0.033	44.52 %	-18.27 %	-10.79 %	4.30 %	1.64 %
	140 Call	0.887	0.663	-0.522	-0.205	0.136	0.030	74.77 %	-58.78 %	-23.16 %	15.35 %	3.36 %
	150 Call	0.372	0.396	-0.548	-0.104	0.189	-0.009	106.35 %	-147.29 %	-27.82 %	50.82 %	-2.34 %
iii	50 Put	0.633	-0.038	0.094	0.061	0.015	0.005	-6.05 %	14.84 %	9.64 %	2.33 %	0.85 %
	60 Put	1.335	-0.063	0.111	0.058	0.013	0.006	-4.74 %	8.32 %	4.34 %	0.97 %	0.42 %
	70 Put	2.571	-0.072	0.121	0.048	0.011	0.006	-2.79 %	4.72 %	1.87 %	0.45 %	0.22 %
	80 Put	4.579	-0.046	0.124	0.034	0.010	0.005	-1.00 %	2.71 %	0.75 %	0.22 %	0.12 %
	90 Put	7.608	0.019	0.119	0.019	0.008	0.004	0.25 %	1.57 %	0.26 %	0.11 %	0.05 %
	100 Call	11.857	0.111	0.111	0.008	0.008	0.004	0.94 %	0.94 %	0.07 %	0.07 %	0.03 %
	110 Call	7.430	0.197	0.096	-0.004	0.008	0.003	2.65 %	1.29 %	-0.05 %	0.10 %	0.05 %
	120 Call	4.289	0.244	0.074	-0.015	0.009	0.004	5.70 %	1.74 %	-0.36 %	0.20 %	0.09 %
	130 Call	2.260	0.239	0.046	-0.027	0.009	0.003	10.57 %	2.03 %	-1.21 %	0.40 %	0.14 %
	140 Call	1.080	0.192	0.017	-0.036	0.009	0.002	17.77 %	1.62 %	-3.30 %	0.88 %	0.19 %
	150 Call	0.466	0.129	-0.004	-0.036	0.010	0.001	27.62 %	-0.75 %	-7.81 %	2.13 %	0.13 %

Table 4: Monte Carlo prices and approximation errors by the log-normal asymptotic expansions in case iv, v, and vi.

			Log Normal A.E.(Difference)					Log Normal A.E.(Relative Difference)				
Case	Strike(C/P)	MC	Log-Norm	1st	2nd	3rd	4th	Log-Norm	1st	2nd	3rd	4th
iv	50 Put	9.429	-0.896	0.250	0.470	-0.223	0.021	-9.51 %	2.65 %	4.99 %	-2.36 %	0.22 %
	60 Put	13.095	-0.187	0.168	0.449	-0.215	0.028	-1.43 %	1.29 %	3.43 %	-1.64 %	0.21 %
	70 Put	17.307	0.678	0.045	0.431	-0.203	0.034	3.92 %	0.26 %	2.49 %	-1.17 %	0.19 %
	80 Put	22.041	1.620	-0.099	0.414	-0.190	0.039	7.35 %	-0.45 %	1.88 %	-0.86 %	0.18 %
	90 Put	27.272	2.577	-0.253	0.397	-0.177	0.045	9.45 %	-0.93 %	1.45 %	-0.65 %	0.17 %
	100 Call	32.971	3.503	-0.416	0.379	-0.163	0.051	10.62 %	-1.26 %	1.15 %	-0.49 %	0.15 %
	110 Call	29.110	4.367	-0.589	0.360	-0.149	0.057	15.00 %	-2.02 %	1.24 %	-0.51 %	0.20 %
	120 Call	25.655	5.149	-0.773	0.338	-0.135	0.063	20.07 %	-3.01 %	1.32 %	-0.53 %	0.25 %
	130 Call	22.576	5.837	-0.972	0.315	-0.120	0.069	25.85 %	-4.30 %	1.39 %	-0.53 %	0.31 %
	140 Call	19.842	6.424	-1.186	0.289	-0.104	0.076	32.38 %	-5.98 %	1.46 %	-0.53 %	0.38 %
	150 Call	17.420	6.912	-1.416	0.261	-0.088	0.083	39.68 %	-8.13 %	1.50 %	-0.50 %	0.47 %
v	50 Put	15.350	0.961	-0.125	0.782	-0.523	0.148	6.26 %	-0.82 %	5.10 %	-3.41 %	0.96 %
	60 Put	20.207	1.990	-0.391	0.823	-0.513	0.153	9.85 %	-1.93 %	4.07 %	-2.54 %	0.76 %
	70 Put	25.499	3.062	-0.664	0.857	-0.495	0.153	12.01 %	-2.60 %	3.36 %	-1.94 %	0.60 %
	80 Put	31.184	4.134	-0.937	0.884	-0.472	0.150	13.26 %	-3.00 %	2.84 %	-1.51 %	0.48 %
	90 Put	37.228	5.175	-1.207	0.908	-0.446	0.145	13.90 %	-3.24 %	2.44 %	-1.20 %	0.39 %
	100 Call	43.598	6.168	-1.474	0.928	-0.417	0.137	14.15 %	-3.38 %	2.13 %	-0.96 %	0.31 %
	110 Call	40.267	7.101	-1.741	0.946	-0.387	0.129	17.63 %	-4.32 %	2.35 %	-0.96 %	0.32 %
	120 Call	37.208	7.967	-2.009	0.962	-0.356	0.119	21.41 %	-5.40 %	2.59 %	-0.96 %	0.32 %
	130 Call	34.399	8.763	-2.278	0.977	-0.323	0.107	25.47 %	-6.62 %	2.84 %	-0.94 %	0.31 %
	140 Call	31.818	9.487	-2.551	0.990	-0.289	0.095	29.82 %	-8.02 %	3.11 %	-0.91 %	0.30 %
	150 Call	29.447	10.142	-2.829	1.003	-0.255	0.082	34.44 %	-9.61 %	3.41 %	-0.87 %	0.28 %
vi	50 Put	19.801	2.280	-0.889	1.143	-0.592	0.182	11.51 %	-4.49 %	5.77 %	-2.99 %	0.92 %
	60 Put	25.471	3.371	-1.248	1.254	-0.581	0.154	13.23 %	-4.90 %	4.93 %	-2.28 %	0.60 %
	70 Put	31.500	4.459	-1.594	1.351	-0.560	0.120	14.15 %	-5.06 %	4.29 %	-1.78 %	0.38 %
	80 Put	37.847	5.520	-1.927	1.437	-0.535	0.081	14.59 %	-5.09 %	3.80 %	-1.41 %	0.21 %
	90 Put	44.476	6.541	-2.246	1.515	-0.505	0.039	14.71 %	-5.05 %	3.41 %	-1.14 %	0.09 %
	100 Call	51.357	7.512	-2.555	1.586	-0.474	-0.005	14.63 %	-4.98 %	3.09 %	-0.92 %	-0.01 %
	110 Call	48.465	8.430	-2.856	1.652	-0.442	-0.051	17.39 %	-5.89 %	3.41 %	-0.91 %	-0.10 %
	120 Call	45.780	9.291	-3.150	1.715	-0.409	-0.098	20.30 %	-6.88 %	3.75 %	-0.89 %	-0.21 %
	130 Call	43.281	10.097	-3.439	1.774	-0.376	-0.147	23.33 %	-7.94 %	4.10 %	-0.87 %	-0.34 %
	140 Call	40.954	10.848	-3.724	1.831	-0.342	-0.197	26.49 %	-9.09 %	4.47 %	-0.84 %	-0.48 %
	150 Call	38.782	11.545	-4.007	1.886	-0.309	-0.248	29.77 %	-10.33 %	4.86 %	-0.80 %	-0.64 %

5.2 Currency Option under a Libor Market Model of Interest Rates and a Stochastic Volatility of a Spot Exchange Rate

In this subsection, we apply our methods to pricing options on currencies under Libor Market Models(LMMs) of interest rates and a stochastic volatility of the spot foreign exchange rate(Forex). Due to limitation of space, only the structure of the stochastic differential equations of our model is described here. For details of the underlying model, see Takahashi and Takehara [53].

5.2.1 Cross-Currency Libor Market Models

Let $(\Omega, \mathcal{F}, \tilde{P}, \{\mathcal{F}_t\}_{0\leq t \leq T^* < \infty})$ be a complete probability space with a filtration satisfying the usual conditions. We consider the following pricing problem for the call option with maturity $T \in (0, T^*]$ and strike rate $K > 0$;

$$V^C(0;T,K) = P_d(0,T) \times \mathbf{E}^P\left[(S(T)-K)^+\right] = P_d(0,T) \times \mathbf{E}^P\left[(F_T(T)-K)^+\right] \tag{41}$$

where $V^C(0;T,K)$ denotes the value of an European call option at time 0 with maturity T and strike rate K, $S(T)$ denotes the spot exchange rate at time $t \geq 0$ and $F_T(t)$ denotes the time t value of the forex forward rate with maturity T. Similarly, for the put option we consider

$$V^P(0;T,K) = P_d(0,T) \times \mathbf{E}^P\left[(K-S(T))^+\right] = P_d(0,T) \times \mathbf{E}^P\left[(K-F_T(T))^+\right]. \tag{42}$$

It is well known that the arbitrage-free relation between the forex spot rate and the forex forward rate are given by $F_T(t) = S(t)\frac{P_f(t,T)}{P_d(t,T)}$ where $P_d(t,T)$ and $P_f(t,T)$ denote the time t values of domestic and foreign zero coupon bonds with maturity T respectively. $\mathbf{E}^P[\cdot]$ denotes an expectation operator under EMM(Equivalent Martingale Measure) P whose associated numeraire is the domestic zero coupon bond maturing at T.

For these pricing problems, a market model and a stochastic volatility model are applied to modeling interest rates' and the spot exchange rate's dynamics respectively.

We first define domestic and foreign forward interest rates as $f_{dj}(t) = \left(\frac{P_d(t,T_j)}{P_d(t,T_{j+1})} - 1\right)\frac{1}{\tau_j}$ and $f_{fj}(t) = \left(\frac{P_f(t,T_j)}{P_f(t,T_{j+1})} - 1\right)\frac{1}{\tau_j}$ respectively, where $j = n(t), n(t)+1, \cdots, N$, $\tau_j = T_{j+1} - T_j$, and $P_d(t,T_j)$ and $P_f(t,T_j)$ denote the prices of domestic/foreign zero coupon bonds with maturity T_j at time $t (\leq T_j)$ respectively; $n(t) = \min\{i : t \leq T_i\}$. We also define spot interest rates to the nearest fixing date denoted by $f_{d,n(t)-1}(t)$ and $f_{f,n(t)-1}(t)$ as $f_{d,n(t)-1}(t) = \left(\frac{1}{P_d(t,T_{n(t)})} - 1\right)\frac{1}{(T_{n(t)}-t)}$ and $f_{f,n(t)-1}(t) = \left(\frac{1}{P_f(t,T_{n(t)})} - 1\right)\frac{1}{(T_{n(t)}-t)}$. Finally, we set $T = T_{N+1}$ and will abbreviate $F_{T_{N+1}}(t)$ to $F_{N+1}(t)$ in what follows.

Under the framework of the asymptotic expansion in the standard cross-currency libor market model, we have to consider the following system of stochastic differential equations(henceforth called S.D.E.s) under the domestic terminal measure P to price options. For detailed arguments on the framework of these S.D.E.s see [53].

As for the domestic and foreign interest rates we assume forward market models; for $j = n(t)-1, n(t), n(t)+1, \cdots, N$,

$$f_{dj}^{(\epsilon)}(t) = f_{dj}(0) + \epsilon^2 \sum_{i=j+1}^{N} \int_0^t g_{di}^{0,(\epsilon)}(u)' \gamma_{dj}(u) f_{dj}^{(\epsilon)}(u) du + \epsilon \int_0^t f_{dj}^{(\epsilon)}(u) \gamma_{dj}'(u) dW_u, \tag{43}$$

$$\begin{aligned} f_{fj}^{(\epsilon)}(t) &= f_{fj}(0) - \epsilon^2 \sum_{i=0}^{j} \int_0^t g_{fi}^{0,(\epsilon)}(u)' \gamma_{fj}(u) f_{fj}^{(\epsilon)}(u) du + \epsilon^2 \sum_{i=0}^{N} \int_0^t g_{di}^{0,(\epsilon)}(u)' \gamma_{fj}(u) f_{fj}^{(\epsilon)}(u) du \\ &\quad - \epsilon^2 \int_0^t \sigma^{(\epsilon)}(u) \bar{\sigma}' \gamma_{fj}(u) f_{fj}^{(\epsilon)}(u) du + \epsilon \int_0^t f_{fj}^{(\epsilon)}(u) \gamma_{fj}'(u) dW_u, \end{aligned} \tag{44}$$

where

$$g_{dj}^{0,(\epsilon)}(t) := \frac{-\tau_j f_{dj}^{(\epsilon)}(t)}{1+\tau_j f_{dj}^{(\epsilon)}(t)} \gamma_{dj}(t), \quad g_{fj}^{0,(\epsilon)}(t) := \frac{-\tau_j f_{fj}^{(\epsilon)}(t)}{1+\tau_j f_{fj}^{(\epsilon)}(t)} \gamma_{fj}(t);$$

x' denotes the transpose of x, $\hat{J}_{j+1} := \{0, 1, \cdots, j\}$, and W is a d'-dimensional standard Wiener process under the domestic terminal measure P; $\gamma_{dj}(s)$, $\gamma_{fj}(s)$ are d'-dimensional vector-valued functions of time-parameter s; $\bar{\sigma}$ denotes a d'-dimensional constant vector satisfying $||\bar{\sigma}|| = 1$ and $\sigma(t)$, the volatility of the spot exchange rate, is specified to follow a $\mathbf{R}_{++}$-valued general time-inhomogeneous Markovian process as follows:

$$\sigma(t) = \sigma(0) + \int_0^t \mu(u, \sigma^{(\epsilon)}(u)) du + \epsilon^2 \sum_{j=1}^{N} \int_0^t g_{dj}^{0,(\epsilon)}(u)' \omega(u, \sigma^{(\epsilon)}(u)) du + \epsilon \int_0^t \omega'(u, \sigma^{(\epsilon)}(u)) dW_u, \tag{45}$$

where $\mu(s, x)$ and $\omega(s, x)$ are functions of s and x.

Finally, we consider the process of the forex forward $F_{N+1}(t)$. Since $F_{N+1}(t) \equiv F_{T_{N+1}}(t)$ can be expressed as $F_{N+1}(t) = S(t) \frac{P_f(t, T_{N+1})}{P_d(t, T_{N+1})}$, we easily notice that it is a martingale under the domestic terminal measure. In particular, it satisfies the following stochastic differential equation

$$F_T^{(\epsilon)}(t) = F_T(0) + \epsilon \int_0^t \sigma_F^{(\epsilon)}(u)' F^{(\epsilon)}(u) dW_u \tag{46}$$

where

$$\sigma_F^{(\epsilon)}(t) := \sum_{j=0}^{N} \left(g_{fj}^{0,(\epsilon)}(t) - g_{dj}^{0,(\epsilon)}(t) \right) + \sigma^{(\epsilon)}(t).$$

5.2.2 Numerical Examples

We here specify our model and parameters, and confirm the effectiveness of our method in this cross-currency framework.

First of all, the processes of domestic and foreign forward interest rates and of the volatility of the spot exchange rate are specified. We suppose $d' = 4$, that is the dimension of a Brownian motion is set to be four; it represents the uncertainty of domestic and foreign interest rates, the spot exchange rate, and its volatility. Note that in this framework correlations among all factors are allowed. We also suppose $S(0) = 100$.

Next, we specify a volatility process of the spot exchange rate in (45) with

$$\begin{cases} \mu(s, x) = \kappa(\theta - x), \\ \omega(s, x) = \omega x, \end{cases} \tag{47}$$

Table 5: Initial domestic/foreign forward interest rates and their volatilities

	f_d	γ_d^*	f_f	γ_f^*
case (i)	0.05	0.12	0.05	0.12
case (ii)	0.02	0.3	0.05	0.12
case (iii)	0.05	0.12	0.02	0.3
case (iv)	0.02	0.3	0.02	0.3

where θ and κ represent the level and speed of its mean-reversion respectively, and ω denotes a volatility vector on the volatility. In this section the parameters are set as follows; $\epsilon = 1$, $\sigma(0) = \theta = 0.1$, and $\kappa = 0.1$; $\omega = \omega^* \bar{v}$ where $\omega^* = 0.3$ and $\bar{v}$ denotes a four dimensional constant vector given below.

We further suppose that initial term structures of domestic and foreign forward interest rates are flat, and their volatilities also have flat structures and are constant over time: that is, for all $j, f_{dj}(0) = f_d$, $f_{fj}(0) = f_f$, $\gamma_{dj}(t) = \gamma_d^* \bar{\gamma}_d 1_{\{t<T_j\}}(t)$ and $\gamma_{fj}(t) = \gamma_f^* \bar{\gamma}_f 1_{\{t<T_j\}}(t)$. Here, γ_d^* and γ_f^* are constant scalars, and $\bar{\gamma}_d$ and $\bar{\gamma}_f$ denote four dimensional constant vectors. Moreover, given a correlation matrix $\underline{C}$ among all four factors, the constant vectors $\bar{\gamma}_d$, $\bar{\gamma}_f$, $\bar{\sigma}$ and $\bar{v}$ can be determined to satisfy $||\bar{\gamma}_d|| = ||\bar{\gamma}_f|| = ||\bar{\sigma}|| = ||\bar{v}|| = 1$ and $V'V = \underline{C}$ where $V := (\bar{\gamma}_d, \bar{\gamma}_f, \bar{\sigma}, \bar{v})$.

In this subsection, we consider four different cases for f_d, γ_d^*, f_f and γ_f^* as in Table 5. For correlations, four sets of parameters are considered: In the case "Corr.1", all the factors are independent: In "Corr.2", there exists only the correlation of -0.5 between the spot exchange rate and its volatility (i.e. $\bar{\sigma}'\bar{v} = -0.5$) while there are no correlations among the others: In "Corr.3", the correlation between interest rates and the spot exchange rate are allowed while there are no correlations among the others; the correlation between domestic ones and the spot forex is 0.5($\bar{\gamma}_d'\bar{\sigma} = 0.5$) and the correlation between foreign ones and the spot forex is -0.5($\bar{\gamma}_f'\bar{\sigma} = -0.5$): Finally in "Corr.4", more intricately correlated structure is considered; $\bar{\gamma}_d'\bar{\sigma} = 0.5$, $\bar{\gamma}_f'\bar{\sigma} = -0.5$ between interest rates and the spot forex; and $\bar{\sigma}'\bar{v} = -0.5$ between the spot forex and its volatility. It is well known that (both of exact and approximate)evaluation of the long-term options is a hard task in the case with complex structures of correlations such as in "Corr.3" or "Corr.4".

Lastly, we make an assumption that $\gamma_{dn(t)-1}(t)$ and $\gamma_{fn(t)-1}(t)$, volatilities of the domestic and foreign interest rates applied to the period from t to the next fixing date $T_{n(t)}$, are equal to be zero for arbitrary $t \in [t, T_{n(t)}]$.

In Table 6-9 and Figure 1, we compare our estimations of the values of call and put options by an asymptotic expansion up to the fourth order to the benchmarks estimated by 10^6 trials of Monte Carlo simulation which is discretized by Euler-Maruyama scheme with time step 0.05 and applied the Antithetic Variable Method. For the moneynesses(defined by $K/F_{N+1}(0)$) less than one, the prices of put options are shown; otherwise, the prices of call options are displayed.

As seen in these tables and figure, in general the estimators show more accuracy as the order of the expansion increases. Especially, for the deep OTM options the fourth order approximation performs much better and is stabler than the approximation with lower orders.

Table 6

Case (i)		Corr.1																			
Moneyness		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
MC		0.0021	0.0136	0.051	0.1504	0.3865	0.8867	1.8303	3.4047	5.7408	8.861	6.5943	4.9199	3.7043	2.8275	2.1939	1.7316	1.3904	1.1351	0.9451	0.7922
A.E.	1st	0.0325	0.0781	0.1749	0.3663	0.7181	1.3209	2.2859	3.7325	5.7694	8.4733	5.7694	3.7324	2.2859	1.3209	0.7181	0.3663	0.1749	0.078	0.0325	0.0126
	2nd	−0.102	−0.163	−0.217	−0.209	−0.037	0.4454	1.4093	3.0141	5.3628	8.4733	6.1759	4.4508	3.1625	2.1964	1.4728	0.9414	0.5673	0.3194	0.1671	0.0809
	3rd	0.2001	0.2571	0.3089	0.386	0.5755	1.0293	1.9409	3.4945	5.8091	8.9081	6.6222	4.9312	3.6941	2.7802	2.085	1.5363	1.0936	0.7398	0.4693	0.2763
	4th	−0.181	−0.129	−0.031	0.1267	0.4012	0.9201	1.8706	3.4461	5.7786	8.8977	6.633	4.9629	3.7517	2.8805	2.2531	1.7907	1.4292	1.1223	0.847	0.6014
Diff.																					
A.E.	1st	0.0304	0.0645	0.1239	0.2159	0.3316	0.4342	0.4556	0.3278	0.0286	−0.388	−0.825	−1.188	−1.418	−1.507	−1.476	−1.365	−1.216	−1.057	−0.913	−0.78
	2nd	−0.104	−0.177	−0.268	−0.359	−0.423	−0.441	−0.421	−0.391	−0.378	−0.388	−0.418	−0.469	−0.542	−0.631	−0.721	−0.79	−0.823	−0.816	−0.778	−0.711
	3rd	0.198	0.2435	0.2579	0.2356	0.189	0.1426	0.1106	0.0898	0.0683	0.0471	0.0279	0.0113	−0.01	−0.047	−0.109	−0.195	−0.297	−0.395	−0.476	−0.516
	4th	−0.183	−0.143	−0.082	−0.024	0.0147	0.0334	0.0403	0.0414	0.0378	0.0367	0.0387	0.043	0.0474	0.053	0.0592	0.0591	0.0388	−0.013	−0.098	−0.191

Case (ii)		Corr.1																			
Moneyness		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
MC		0.0021	0.0142	0.0526	0.1534	0.3897	0.8898	1.8359	3.4182	5.7667	8.9083	6.6647	5.0085	3.8038	2.9298	2.2932	1.8239	1.4742	1.2101	1.0074	0.8502
A.E.	1st	0.0329	0.0789	0.1764	0.3688	0.7221	1.3266	2.2935	3.7418	5.78	8.4844	5.78	3.7418	2.2935	1.3266	0.722	0.3688	0.1764	0.0788	0.0329	0.0128
	2nd	−0.107	−0.172	−0.231	−0.227	−0.059	0.4214	1.3878	3	5.3603	8.4844	6.1996	4.4836	3.1992	2.2318	1.5032	0.9648	0.5835	0.3297	0.173	0.084
	3rd	0.2151	0.2758	0.3296	0.4068	0.5957	1.0506	1.9676	3.5318	5.8608	8.9744	6.7001	5.0153	3.7789	2.861	2.1579	1.5987	1.1439	0.7775	0.4954	0.2929
	4th	−0.197	−0.142	−0.04	0.1192	0.3925	0.9108	1.8666	3.4573	5.8147	8.9631	6.7246	5.0714	3.8655	2.99	2.3531	1.8798	1.5075	1.1897	0.9026	0.6444
Diff.																					
A.E.	1st	0.0308	0.0647	0.1238	0.2154	0.3324	0.4368	0.4576	0.3236	0.0133	−0.424	−0.885	−1.267	−1.51	−1.603	−1.571	−1.455	−1.298	−1.131	−0.975	−0.837
	2nd	−0.109	−0.186	−0.283	−0.38	−0.449	−0.468	−0.448	−0.418	−0.406	−0.424	−0.465	−0.525	−0.605	−0.698	−0.79	−0.859	−0.891	−0.88	−0.834	−0.766
	3rd	0.213	0.2616	0.277	0.2534	0.206	0.1608	0.1317	0.1136	0.0941	0.0661	0.0354	0.0068	−0.025	−0.069	−0.135	−0.225	−0.33	−0.433	−0.512	−0.557
	4th	−0.199	−0.156	−0.093	−0.034	0.0028	0.021	0.0307	0.0391	0.048	0.0548	0.0599	0.0629	0.0617	0.0602	0.0599	0.0559	0.0333	−0.02	−0.105	−0.206

Case (iii)		Corr.1																			
Moneyness		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
MC		0.0031	0.0205	0.0776	0.2271	0.5707	1.2768	2.5725	4.6977	7.8173	11.978	8.9033	6.6303	4.9801	3.7907	2.9308	2.3056	1.8445	1.5008	1.2392	1.0365
A.E.	1st	0.0442	0.1059	0.2369	0.4953	0.9697	1.7815	3.08	5.0249	7.7621	11.394	7.762	5.0249	3.08	1.7815	0.9696	0.4952	0.2369	0.1058	0.0441	0.0172
	2nd	−0.133	−0.211	−0.277	−0.257	−0.017	0.6386	1.9365	4.0883	7.2322	11.394	8.2919	5.9615	4.2235	2.9244	1.9559	1.2477	0.7509	0.4226	0.2211	0.1071
	3rd	0.2663	0.348	0.4292	0.5517	0.8304	1.4644	2.707	4.8005	7.9041	12.052	8.9638	6.6737	4.994	3.7502	2.8029	2.0566	1.4573	0.9815	0.6202	0.364
	4th	−0.249	−0.182	−0.044	0.1866	0.5865	1.3198	2.6254	4.7519	7.8739	12.037	8.9651	6.6975	5.0565	3.8811	3.0373	2.4145	1.9246	1.5062	1.1312	0.7989
Diff.																					
A.E.	1st	0.0411	0.0854	0.1593	0.2682	0.399	0.5047	0.5075	0.3272	−0.055	−0.584	−1.141	−1.605	−1.9	−2.009	−1.961	−1.81	−1.608	−1.395	−1.195	−1.019
	2nd	−0.136	−0.231	−0.355	−0.484	−0.587	−0.638	−0.636	−0.609	−0.585	−0.584	−0.611	−0.669	−0.757	−0.866	−0.975	−1.058	−1.094	−1.078	−1.018	−0.929
	3rd	0.2632	0.3275	0.3516	0.3246	0.2597	0.1876	0.1345	0.1028	0.0868	0.0738	0.0605	0.0434	0.0139	−0.041	−0.128	−0.249	−0.387	−0.519	−0.619	−0.673
	4th	−0.252	−0.203	−0.122	−0.041	0.0158	0.043	0.0529	0.0542	0.0566	0.0586	0.0618	0.0672	0.0764	0.0904	0.1065	0.1089	0.0801	0.0054	−0.108	−0.238

Case (iv)		Corr.1																			
Moneyness		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
MC		0.0032	0.0202	0.0771	0.2279	0.5732	1.2829	2.5829	4.7165	7.8575	12.047	9.0009	6.7503	5.1155	3.9308	3.069	2.4361	1.9648	1.6092	1.3368	1.1245
A.E.	1st	0.0447	0.107	0.2389	0.4987	0.975	1.7892	3.0902	5.0375	7.7763	11.409	7.7763	5.0375	3.0902	1.7892	0.9749	0.4986	0.2389	0.1069	0.0447	0.0174
	2nd	−0.14	−0.223	−0.295	−0.282	−0.047	0.6064	1.9077	4.0695	7.2288	11.409	8.3237	6.0056	4.2727	2.972	1.9966	1.279	0.7727	0.4364	0.2291	0.1113
	3rd	0.2864	0.3732	0.4574	0.5802	0.8582	1.4935	2.743	4.8501	7.9726	12.14	9.0675	6.7862	5.108	3.8592	2.9016	2.1409	1.5251	1.0322	0.6552	0.3862
	4th	−0.271	−0.199	−0.056	0.1764	0.5752	1.3079	2.6203	4.7667	7.9215	12.124	9.0872	6.8426	5.2092	4.0286	3.1724	2.5351	2.0306	1.5973	1.2061	0.8566
Diff.																					
A.E.	1st	0.0415	0.0868	0.1618	0.2708	0.4018	0.5063	0.5073	0.321	−0.081	−0.638	−1.225	−1.713	−2.025	−2.142	−2.094	−1.938	−1.726	−1.502	−1.292	−1.107
	2nd	−0.143	−0.243	−0.372	−0.51	−0.62	−0.677	−0.675	−0.647	−0.629	−0.638	−0.677	−0.745	−0.843	−0.959	−1.072	−1.157	−1.192	−1.173	−1.108	−1.013
	3rd	0.2832	0.353	0.3803	0.3523	0.285	0.2106	0.1601	0.1336	0.1151	0.093	0.0666	0.0359	−0.008	−0.072	−0.167	−0.295	−0.44	−0.577	−0.682	−0.738
	4th	−0.274	−0.219	−0.133	−0.051	0.002	0.025	0.0374	0.0502	0.064	0.0766	0.0863	0.0923	0.0937	0.0978	0.1034	0.099	0.0658	−0.012	−0.131	−0.268

Table 7

Case (i)		Corr.2																			
Moneyness		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
MC		0.0065	0.0366	0.1137	0.2784	0.5955	1.1579	2.0956	3.5506	5.6542	8.4874	5.9496	4.0731	2.7427	1.8319	1.2232	0.8223	0.5598	0.3875	0.2736	0.1975
A.E.	1st	0.0325	0.0781	0.1749	0.3663	0.7181	1.3209	2.2859	3.7325	5.7694	8.4733	5.7694	3.7324	2.2859	1.3209	0.7181	0.3663	0.1749	0.078	0.0325	0.0126
	2nd	######	0.0182	0.0777	0.2238	0.5311	1.104	2.0687	3.5545	5.6686	8.4733	5.8701	3.9104	2.5031	1.5378	0.9051	0.5088	0.2722	0.1378	0.0658	0.0295
	3rd	0.0853	0.1526	0.266	0.4596	0.7925	1.3576	2.2812	3.7078	5.7701	8.5543	5.9716	4.0638	2.7156	1.7914	1.1665	0.7446	0.4605	0.2721	0.152	0.0794
	4th	0.0149	0.0627	0.1598	0.3408	0.665	1.2264	2.1559	3.6041	5.7064	8.5437	6.0151	4.1499	2.8269	1.912	1.2862	0.8576	0.5625	0.3588	0.2201	0.1282
Diff.																					
A.E.	1st	0.026	0.0415	0.0612	0.0879	0.1226	0.163	0.1903	0.1819	0.1152	−0.014	−0.18	−0.341	−0.457	−0.511	−0.505	−0.456	−0.385	−0.31	−0.241	−0.185
	2nd	−0.007	−0.018	−0.036	−0.055	−0.064	−0.054	−0.027	0.0039	0.0144	−0.014	−0.08	−0.163	−0.24	−0.294	−0.318	−0.314	−0.288	−0.25	−0.208	−0.168
	3rd	0.0788	0.116	0.1523	0.1812	0.197	0.1997	0.1856	0.1572	0.1159	0.0669	0.022	−0.009	−0.027	−0.041	−0.057	−0.078	−0.099	−0.115	−0.122	−0.118
	4th	0.0084	0.0261	0.0461	0.0624	0.0695	0.0685	0.0603	0.0535	0.0522	0.0563	0.0655	0.0768	0.0842	0.0801	0.063	0.0353	0.0027	−0.029	−0.054	−0.069

Case (ii)		Corr.2																			
Moneyness		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
MC		0.0063	0.0361	0.114	0.2801	0.5966	1.1597	2.0998	3.5616	5.6763	8.5259	6.0095	4.1544	2.8405	1.9379	1.3285	0.9215	0.6489	0.4641	0.3376	0.2497
A.E.	1st	0.0329	0.0789	0.1764	0.3688	0.7221	1.3266	2.2935	3.7418	5.78	8.4844	5.78	3.7418	2.2935	1.3266	0.722	0.3688	0.1764	0.0788	0.0329	0.0128
	2nd	−0.005	0.0101	0.0648	0.2055	0.508	1.0786	2.0453	3.5385	5.665	8.4844	5.895	3.9451	2.5417	1.5747	0.9361	0.5321	0.288	0.1475	0.0713	0.0323
	3rd	0.0983	0.1716	0.291	0.489	0.8238	1.3882	2.3105	3.7381	5.8067	8.6031	6.0367	4.1447	2.8069	1.8844	1.2519	0.8157	0.5142	0.309	0.1751	0.0926
	4th	−0.002	0.0439	0.1412	0.3246	0.6523	1.2179	2.1532	3.6116	5.7307	8.5918	6.091	4.2521	2.9482	2.042	1.4134	0.9722	0.6579	0.4322	0.2721	0.1621
Diff.																					
A.E.	1st	0.0266	0.0428	0.0624	0.0887	0.1255	0.1669	0.1937	0.1802	0.1037	−0.041	−0.23	−0.413	−0.547	−0.611	−0.607	−0.553	−0.473	−0.385	−0.305	−0.237
	2nd	−0.012	−0.026	−0.049	−0.075	−0.089	−0.081	−0.055	−0.023	−0.011	−0.041	−0.115	−0.209	−0.299	−0.363	−0.392	−0.389	−0.361	−0.317	−0.266	−0.217
	3rd	0.092	0.1355	0.177	0.2089	0.2272	0.2285	0.2107	0.1765	0.1304	0.0772	0.0272	−0.01	−0.034	−0.053	−0.077	−0.106	−0.135	−0.155	−0.163	−0.157
	4th	−0.008	0.0078	0.0272	0.0445	0.0557	0.0582	0.0534	0.05	0.0544	0.0659	0.0815	0.0977	0.1077	0.1041	0.0849	0.0507	0.009	−0.032	−0.066	−0.088

Case (iii)		Corr.2																			
Moneyness		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
MC		0.0092	0.0513	0.1611	0.3953	0.8444	1.6351	2.93	4.9079	7.7376	11.521	8.0786	5.5251	3.7142	2.476	1.6485	1.1038	0.7472	0.5144	0.361	0.2583
A.E.	1st	0.0442	0.1059	0.2369	0.4953	0.9697	1.7815	3.08	5.0249	7.7621	11.394	7.762	5.0249	3.08	1.7815	0.9696	0.4952	0.2369	0.1058	0.0441	0.0172
	2nd	0.0039	0.0337	0.1198	0.3239	0.7449	1.521	2.8194	4.8115	7.6683	11.394	7.8828	5.2384	3.3406	2.042	1.1944	0.6667	0.354	0.178	0.0845	0.0377
	3rd	0.1137	0.2071	0.3669	0.6403	1.1075	1.8916	3.1576	5.0929	7.87	11.601	8.1114	5.5198	3.6788	2.4125	1.557	0.9832	0.6011	0.3514	0.1943	0.1007
	4th	0.0182	0.0824	0.2172	0.473	0.9317	1.7175	2.9984	4.9654	7.792	11.585	8.159	5.6209	3.817	2.571	1.7216	1.1425	0.7452	0.4722	0.2872	0.1656
Diff.																					
A.E.	1st	0.035	0.0546	0.0758	0.1	0.1253	0.1464	0.15	0.117	0.0245	−0.127	−0.317	−0.5	−0.634	−0.695	−0.679	−0.609	−0.51	−0.409	−0.317	−0.241
	2nd	−0.005	−0.018	−0.041	−0.071	−0.099	−0.114	−0.111	−0.096	−0.069	−0.127	−0.196	−0.287	−0.374	−0.434	−0.454	−0.437	−0.393	−0.336	−0.277	−0.221
	3rd	0.1045	0.1558	0.2058	0.245	0.2631	0.2565	0.2276	0.185	0.1324	0.0801	0.0328	−0.005	−0.035	−0.063	−0.092	−0.121	−0.146	−0.163	−0.167	−0.158
	4th	0.009	0.0311	0.0561	0.0777	0.0873	0.0824	0.0684	0.0575	0.0544	0.0641	0.0804	0.0958	0.1028	0.095	0.0731	0.0387	−0.002	−0.042	−0.074	−0.093

Case (iv)		Corr.2																			
Moneyness		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
MC		0.0087	0.05	0.1589	0.3929	0.8411	1.6312	2.9244	4.9046	7.7428	11.548	8.136	5.6174	3.8329	2.6083	1.7833	1.233	0.8668	0.6217	0.4539	0.3375
A.E.	1st	0.0447	0.107	0.2389	0.4987	0.975	1.7892	3.0902	5.0375	7.7763	11.409	7.7763	5.0375	3.0902	1.7892	0.9749	0.4986	0.2389	0.1069	0.0447	0.0174
	2nd	−0.002	0.0227	0.1025	0.2993	0.7138	1.4869	2.788	4.7901	7.6364	11.409	7.9162	5.2849	3.3924	2.0915	1.236	0.6981	0.3753	0.1911	0.0918	0.0414
	3rd	0.1312	0.2328	0.4007	0.6803	1.1502	1.9332	3.197	5.1332	7.9181	11.666	8.1979	5.628	3.8015	2.5378	1.6724	1.0791	0.6736	0.4011	0.2254	0.1184
	4th	−0.005	0.0566	0.1915	0.4506	0.9145	1.7067	2.996	4.9766	7.825	11.649	8.259	5.7561	3.9789	2.7457	1.8937	1.2981	0.8747	0.5714	0.3572	0.2109
Diff.																					
A.E.	1st	0.036	0.057	0.08	0.1058	0.1339	0.158	0.1658	0.1329	0.0335	−0.14	−0.36	−0.58	−0.743	−0.819	−0.808	−0.734	−0.628	−0.515	−0.409	−0.32
	2nd	−0.011	−0.027	−0.056	−0.094	−0.127	−0.144	−0.136	−0.115	−0.106	−0.14	−0.22	−0.333	−0.441	−0.517	−0.547	−0.535	−0.492	−0.431	−0.362	−0.296
	3rd	0.1225	0.1828	0.2418	0.2874	0.3091	0.302	0.2726	0.2286	0.1753	0.1174	0.0619	0.0106	−0.031	−0.071	−0.111	−0.154	−0.193	−0.221	−0.229	−0.219
	4th	−0.013	0.0066	0.0326	0.0577	0.0734	0.0755	0.0716	0.072	0.0822	0.1007	0.123	0.1387	0.146	0.1374	0.1104	0.0651	0.0079	−0.05	−0.097	−0.127

Table 8

Case (i)		Corr.3																			
Moneyness		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
MC		0.0027	0.0209	0.0845	0.2576	0.6535	1.4218	2.7132	4.6397	7.2473	10.515	8.2679	6.5157	5.1621	4.12	3.3181	2.6992	2.2193	1.8439	1.5475	1.3115
A.E.	1st	0.148	0.2812	0.5103	0.8852	1.4704	2.3415	3.5808	5.268	7.4705	10.234	7.4705	5.268	3.5807	2.3415	1.4703	0.8852	0.5102	0.2812	0.1478	0.0742
	2nd	-0.317	-0.388	-0.385	-0.223	0.2094	1.0401	2.3909	4.3541	6.9731	10.234	7.9679	6.1818	4.7706	3.6429	2.7313	1.9936	1.4053	0.9503	0.6131	0.3762
	3rd	0.3084	0.294	0.2991	0.4108	0.7602	1.5013	2.7796	4.6979	7.2963	10.552	8.2911	6.5256	5.1594	4.1041	3.2821	2.6276	2.0892	1.6323	1.2388	0.9042
	4th	-0.073	0.0023	0.097	0.2787	0.6712	1.4339	2.7229	4.6503	7.2603	10.53	8.2853	6.5353	5.1839	4.1458	3.3509	2.7436	2.2774	1.9111	1.6074	1.3374
Diff.																					
A.E.	1st	0.1453	0.2603	0.4258	0.6276	0.8169	0.9197	0.8676	0.6283	0.2232	-0.281	-0.797	-1.248	-1.581	-1.779	-1.848	-1.814	-1.709	-1.563	-1.4	-1.237
	2nd	-0.32	-0.409	-0.469	-0.481	-0.444	-0.382	-0.322	-0.286	-0.274	-0.281	-0.3	-0.334	-0.392	-0.477	-0.587	-0.706	-0.814	-0.894	-0.934	-0.935
	3rd	0.3057	0.2731	0.2146	0.1532	0.1067	0.0795	0.0664	0.0582	0.049	0.037	0.0232	0.0099	-0.003	-0.016	-0.036	-0.072	-0.13	-0.212	-0.309	-0.407
	4th	-0.076	-0.019	0.0125	0.0211	0.0177	0.0121	0.0097	0.0106	0.013	0.0154	0.0174	0.0196	0.0218	0.0258	0.0328	0.0444	0.0581	0.0672	0.0599	0.0259

Case (ii)		Corr.3																			
Moneyness		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
MC		0.0027	0.0201	0.0821	0.2509	0.6367	1.3919	2.6766	4.6112	7.2423	10.541	8.3293	6.6101	5.2844	4.2629	3.4733	2.8596	2.3782	1.9985	1.6961	1.4529
A.E.	1st	0.1501	0.2846	0.5155	0.8928	1.4808	2.3549	3.5971	5.2868	7.491	10.255	7.491	5.2868	3.5971	2.3549	1.4807	0.8928	0.5155	0.2846	0.15	0.0755
	2nd	-0.358	-0.445	-0.458	-0.311	0.1128	0.9446	2.3087	4.2978	6.9529	10.255	8.0291	6.2757	4.8856	3.7653	2.8487	2.0968	1.4893	1.0138	0.658	0.4059
	3rd	0.3816	0.3594	0.3452	0.4292	0.7493	1.4682	2.7385	4.6669	7.292	10.585	8.3682	6.6448	5.3154	4.2889	3.4853	2.8372	2.2927	1.8178	1.3975	1.0313
	4th	-0.085	0.0123	0.1162	0.2903	0.6627	1.4037	2.6816	4.6166	7.2535	10.564	8.3645	6.6566	5.339	4.3257	3.5492	2.9569	2.504	2.148	1.8477	1.5703
Diff.																					
A.E.	1st	0.1474	0.2645	0.4334	0.6419	0.8441	0.963	0.9205	0.6756	0.2487	-0.287	-0.838	-1.323	-1.687	-1.908	-1.993	-1.967	-1.863	-1.714	-1.546	-1.377
	2nd	-0.361	-0.465	-0.54	-0.562	-0.524	-0.447	-0.368	-0.313	-0.289	-0.287	-0.3	-0.334	-0.399	-0.498	-0.625	-0.763	-0.889	-0.985	-1.038	-1.047
	3rd	0.3789	0.3393	0.2631	0.1783	0.1126	0.0763	0.0619	0.0557	0.0497	0.0442	0.0389	0.0347	0.031	0.026	0.012	-0.022	-0.086	-0.181	-0.299	-0.422
	4th	-0.088	-0.008	0.0341	0.0394	0.026	0.0118	0.005	0.0054	0.0112	0.0224	0.0352	0.0465	0.0546	0.0628	0.0759	0.0973	0.1258	0.1495	0.1516	0.1174

Case (iii)		Corr.3																			
Moneyness		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
MC		0.0049	0.0374	0.1471	0.424	1.0083	2.0787	3.8183	6.3764	9.8256	14.154	11.085	8.6932	6.8513	5.4403	4.3571	3.5245	2.8796	2.3778	1.9844	1.6731
A.E.	1st	0.2016	0.3822	0.6924	1.199	1.9886	3.1624	4.8306	7.0997	10.06	13.771	10.06	7.0997	4.831	3.1624	1.9885	1.199	0.6923	0.3822	0.2014	0.1014
	2nd	-0.38	-0.453	-0.423	-0.18	0.422	1.5473	3.3552	5.9672	9.4436	13.771	10.676	8.2322	6.3061	4.7775	3.5551	2.5778	1.8047	1.2173	0.7832	0.4798
	3rd	0.4065	0.4188	0.47	0.6696	1.1815	2.2002	3.9131	6.4584	9.8987	14.215	11.131	8.7235	6.864	5.4304	4.3146	3.4271	2.7001	2.0889	1.5699	1.1353
	4th	-0.137	-0.023	0.1426	0.4431	1.0283	2.0924	3.8318	6.3952	9.8518	14.186	11.121	8.7349	6.9007	5.5019	4.4392	3.6309	3.0086	2.5136	2.0971	1.7249
Diff.																					
A.E.	1st	0.1967	0.3448	0.5453	0.775	0.9803	1.0837	1.0123	0.7233	0.2343	-0.383	-1.025	-1.594	-2.02	-2.278	-2.369	-2.326	-2.187	-1.996	-1.783	-1.572
	2nd	-0.385	-0.49	-0.57	-0.604	-0.586	-0.531	-0.463	-0.409	-0.382	-0.383	-0.409	-0.461	-0.545	-0.663	-0.802	-0.947	-1.075	-1.161	-1.201	-1.193
	3rd	0.4016	0.3814	0.3229	0.2456	0.1732	0.1215	0.0948	0.082	0.0731	0.0612	0.0466	0.0303	0.0127	-0.01	-0.042	-0.097	-0.18	-0.289	-0.415	-0.538
	4th	-0.142	-0.06	-0.005	0.0191	0.02	0.0137	0.0135	0.0188	0.0262	0.0319	0.0367	0.0417	0.0494	0.0616	0.0821	0.1064	0.129	0.1358	0.1127	0.0518

Case (iv)		Corr.3																			
Moneyness		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
MC		0.0048	0.0361	0.1429	0.4129	0.9836	2.0364	3.7688	6.3345	9.8133	14.189	11.174	8.8387	7.0468	5.6752	4.6197	3.8027	3.1649	2.6627	2.2626	1.9413
A.E.	1st	0.2045	0.3869	0.6995	1.2093	2.0025	3.1804	4.8526	7.125	10.087	13.8	10.087	7.125	4.8525	3.1804	2.0025	1.2093	0.6994	0.3869	0.2043	0.103
	2nd	-0.435	-0.529	-0.521	-0.298	0.2923	1.419	3.2447	5.8916	9.4164	13.8	10.758	8.3584	6.4604	4.9418	3.7127	2.7164	1.9202	1.3026	0.8436	0.5198
	3rd	0.5061	0.51	0.5373	0.7009	1.173	2.1592	3.8572	6.4112	9.8835	14.25	11.225	8.878	7.0728	5.682	4.5934	3.7151	2.9787	2.3415	1.7844	1.306
	4th	-0.16	-0.014	0.1678	0.4623	1.0231	2.0574	3.7778	6.3458	9.8339	14.221	11.218	8.8909	7.1061	5.745	4.7111	3.9264	3.324	2.8427	2.4288	2.0431
Diff.																					
A.E.	1st	0.1997	0.3508	0.5566	0.7964	1.0189	1.144	1.0838	0.7905	0.274	-0.389	-1.087	-1.714	-2.194	-2.495	-2.617	-2.593	-2.466	-2.276	-2.058	-1.838
	2nd	-0.44	-0.565	-0.664	-0.711	-0.691	-0.617	-0.524	-0.443	-0.397	-0.389	-0.416	-0.48	-0.586	-0.733	-0.907	-1.086	-1.245	-1.36	-1.419	-1.422
	3rd	0.5013	0.4739	0.3944	0.288	0.1894	0.1228	0.0884	0.0767	0.0702	0.0613	0.0513	0.0393	0.026	0.0068	-0.026	-0.088	-0.186	-0.321	-0.478	-0.635
	4th	-0.165	-0.05	0.0249	0.0494	0.0395	0.021	0.009	0.0113	0.0206	0.032	0.0439	0.0522	0.0593	0.0698	0.0914	0.1237	0.1591	0.18	0.1662	0.1018

Table 9

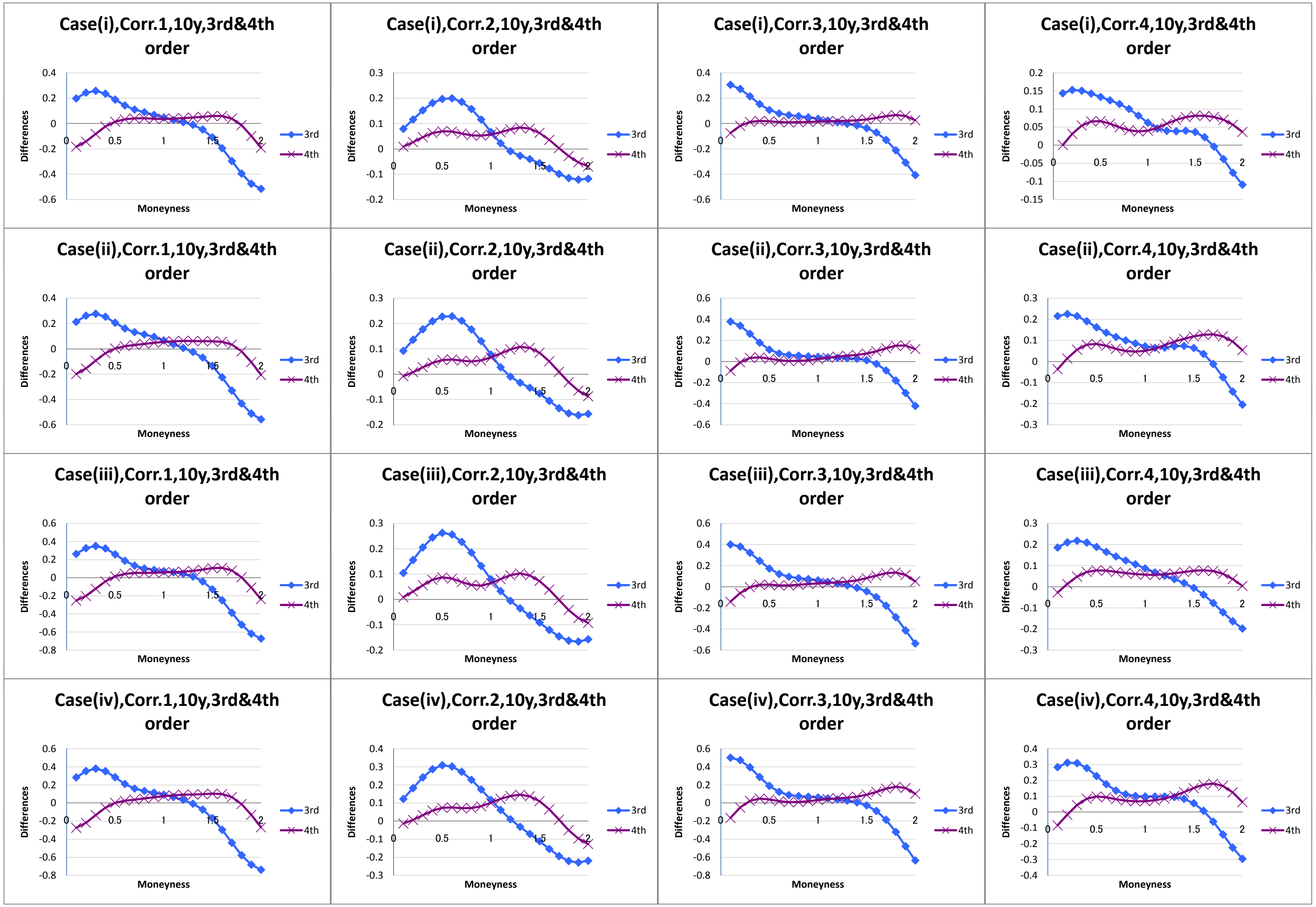
Case (i)		Corr.4																			
Moneyness		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
MC		0.008	0.0478	0.1569	0.3962	0.8563	1.6506	2.8984	4.7016	7.1253	10.182	7.7353	5.8173	4.3474	3.2373	2.4089	1.7966	1.3448	1.0114	0.7649	0.5809
A.E.	1st	0.0008	0.0038	0.0148	0.0504	0.1506	0.3971	0.9292	1.9422	3.6525	6.2344	3.6525	1.9422	0.9292	0.397	0.1505	0.0503	0.0148	0.0038	0.0009	0.0001
	2nd	−0.005	−0.015	−0.037	−0.07	−0.081	0.0289	0.4584	1.4826	3.3637	6.2344	3.9414	2.4017	1.4	0.7651	0.3822	0.1705	0.0668	0.0228	0.0067	0.0017
	3rd	0.1517	0.2005	0.3075	0.5388	0.9896	1.7746	3.0122	4.802	7.2072	10.244	7.7815	5.8571	4.386	3.2773	2.4455	1.8187	1.341	0.9731	0.6888	0.4719
	4th	0.0085	0.0794	0.2113	0.4622	0.923	1.7101	2.9479	4.7434	7.1633	10.222	7.783	5.8772	4.4173	3.3149	2.4906	1.878	1.423	1.0821	0.8215	0.6177
Diff.																					
A.E.	1st	−0.007	−0.044	−0.142	−0.346	−0.706	−1.254	−1.969	−2.759	−3.473	−3.948	−4.083	−3.875	−3.418	−2.84	−2.258	−1.746	−1.33	−1.008	−0.764	−0.581
	2nd	−0.013	−0.063	−0.194	−0.466	−0.937	−1.622	−2.44	−3.219	−3.762	−3.948	−3.794	−3.416	−2.947	−2.472	−2.027	−1.626	−1.278	−0.989	−0.758	−0.579
	3rd	0.1437	0.1527	0.1506	0.1426	0.1333	0.124	0.1138	0.1004	0.0819	0.0622	0.0462	0.0398	0.0386	0.04	0.0366	0.0221	−0.004	−0.038	−0.076	−0.109
	4th	0.0005	0.0316	0.0544	0.066	0.0667	0.0595	0.0495	0.0418	0.038	0.0403	0.0477	0.0599	0.0699	0.0776	0.0817	0.0814	0.0782	0.0707	0.0566	0.0368

Case (ii)		Corr.4																			
Moneyness		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
MC		0.0081	0.0469	0.1531	0.387	0.8384	1.6206	2.8589	4.6608	7.0958	10.181	7.7743	5.9033	4.4778	3.4072	2.6084	2.0139	1.5708	1.2384	0.9868	0.7954
A.E.	1st	0.0008	0.0038	0.0147	0.0501	0.1499	0.3957	0.9269	1.9388	3.6483	6.2299	3.6483	1.9388	0.9269	0.3956	0.1498	0.005	0.0147	0.0038	0.0008	0.0001
	2nd	−0.005	−0.015	−0.038	−0.071	−0.085	0.0225	0.4491	1.4722	3.3549	6.2299	3.9416	2.4054	1.4046	0.7689	0.3845	0.1716	0.0673	0.0229	0.0067	0.0017
	3rd	0.2233	0.2721	0.3683	0.5781	1.0005	1.7572	2.9748	4.7605	7.1813	10.253	7.8391	5.9693	4.5497	3.4811	2.6726	2.0499	1.5586	1.1635	0.8442	0.5905
	4th	−0.029	0.0612	0.2091	0.4663	0.9216	1.6949	2.919	4.7102	7.1422	10.231	7.8355	5.9805	4.5715	3.5145	2.7272	2.1409	1.6994	1.3576	1.0816	0.849
Diff.																					
A.E.	1st	−0.007	−0.043	−0.138	−0.337	−0.689	−1.225	−1.932	−2.722	−3.448	−3.951	−4.126	−3.965	−3.551	−3.012	−2.459	−2.009	−1.556	−1.235	−0.986	−0.795
	2nd	−0.013	−0.062	−0.191	−0.459	−0.923	−1.598	−2.41	−3.189	−3.741	−3.951	−3.833	−3.498	−3.073	−2.638	−2.224	−1.842	−1.504	−1.216	−0.98	−0.794
	3rd	0.2152	0.2252	0.2152	0.1911	0.1621	0.1366	0.1159	0.0997	0.0855	0.0723	0.0648	0.066	0.0719	0.0739	0.0642	0.036	−0.012	−0.075	−0.143	−0.205
	4th	−0.037	0.0143	0.056	0.0793	0.0832	0.0743	0.0601	0.0494	0.0464	0.0503	0.0612	0.0772	0.0937	0.1073	0.1188	0.127	0.1286	0.1192	0.0948	0.0536

Case (iii)		Corr.4																			
Moneyness		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
MC		0.0127	0.076	0.2452	0.6053	1.2708	2.3779	4.0731	6.484	9.7001	13.751	10.409	7.789	5.7858	4.2827	3.1695	2.3523	1.7541	1.3161	0.9951	0.7588
A.E.	1st	0.0011	0.0051	0.0198	0.0673	0.2013	0.5313	1.2447	2.6036	4.8994	8.3662	4.8993	2.6036	1.2447	0.5313	0.2012	0.0672	0.0197	0.0051	0.0011	0.0002
	2nd	−0.007	−0.02	−0.049	−0.091	−0.104	0.0464	0.624	1.9974	4.5182	8.3662	5.2805	3.2099	1.8654	1.0163	0.5061	0.2252	0.088	0.0299	0.0088	0.0022
	3rd	0.1987	0.287	0.4639	0.8148	1.4597	2.5435	4.2166	6.6089	9.8067	13.838	10.477	7.8412	5.8221	4.3008	3.1642	2.315	1.6771	1.1953	0.8315	0.5605
	4th	−0.015	0.089	0.2934	0.6755	1.3486	2.4543	4.1435	6.5492	9.7608	13.808	10.465	7.848	5.8496	4.3525	3.2453	2.4306	1.8286	1.3775	1.0322	0.7629
Diff.																					
A.E.	1st	−0.012	−0.071	−0.225	−0.538	−1.07	−1.847	−2.828	−3.88	−4.801	−5.385	−5.509	−5.185	−4.541	−3.751	−2.968	−2.285	−1.734	−1.311	−0.994	−0.759
	2nd	−0.019	−0.096	−0.294	−0.696	−1.374	−2.332	−3.449	−4.487	−5.182	−5.385	−5.128	−4.579	−3.92	−3.266	−2.663	−2.127	−1.666	−1.286	−0.986	−0.757
	3rd	0.186	0.211	0.2187	0.2095	0.1889	0.1656	0.1435	0.1249	0.1066	0.0873	0.0686	0.0522	0.0363	0.0181	−0.005	−0.037	−0.077	−0.121	−0.164	−0.198
	4th	−0.027	0.013	0.0482	0.0702	0.0778	0.0764	0.0704	0.0652	0.0607	0.0573	0.0564	0.059	0.0638	0.0698	0.0758	0.0783	0.0745	0.0614	0.0371	0.0041

Case (iv)		Corr.4																			
Moneyness		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
MC		0.0123	0.0741	0.2413	0.596	1.2523	2.3474	4.0298	6.4348	9.66	13.74	10.447	7.8881	5.9458	4.4945	3.4194	2.6239	2.0349	1.596	1.266	1.0148
A.E.	1st	0.0011	0.005	0.0196	0.0669	0.2004	0.5295	1.2416	2.5991	4.8937	8.36	4.8936	2.5991	1.2416	0.5295	0.2003	0.0668	0.0196	0.005	0.0011	0.0002
	2nd	−0.007	−0.02	−0.049	−0.093	−0.109	0.0377	0.6116	1.9834	4.5064	8.36	5.2808	3.2148	1.8717	1.0213	0.5092	0.2266	0.0886	0.0301	0.0088	0.0022
	3rd	0.2962	0.3862	0.5509	0.8741	1.4804	2.5236	4.1657	6.5476	9.7624	13.839	10.545	7.9863	6.0411	4.5781	3.4752	2.632	1.9747	1.4544	1.0416	0.7198
	4th	−0.072	0.0576	0.2856	0.6804	1.3509	2.441	4.1112	6.5068	9.7281	13.809	10.521	7.9736	6.0486	4.6204	3.5707	2.7969	2.215	1.7612	1.3908	1.0778
Diff.																					
A.E.	1st	−0.011	−0.069	−0.222	−0.529	−1.052	−1.818	−2.788	−3.836	−4.766	−5.38	−5.553	−5.289	−4.704	−3.965	−3.219	−2.557	−2.015	−1.591	−1.265	−1.015
	2nd	−0.019	−0.094	−0.291	−0.689	−1.361	−2.31	−3.418	−4.451	−5.154	−5.38	−5.166	−4.673	−4.074	−3.473	−2.91	−2.397	−1.946	−1.566	−1.257	−1.013
	3rd	0.2839	0.3121	0.3096	0.2781	0.2281	0.1762	0.1359	0.1128	0.1024	0.0988	0.0981	0.0982	0.0953	0.0836	0.0558	0.0081	−0.06	−0.142	−0.224	−0.295
	4th	−0.084	−0.016	0.0443	0.0844	0.0986	0.0936	0.0814	0.072	0.0681	0.069	0.0746	0.0855	0.1028	0.1259	0.1513	0.173	0.1801	0.1652	0.1248	0.063

Figure 1


Case(i),Corr.1,10y,3rd&4th order
Case(i),Corr.2,10y,3rd&4th order
Case(i),Corr.3,10y,3rd&4th order
Case(i),Corr.4,10y,3rd&4th order
Case(ii),Corr.1,10y,3rd&4th order
Case(ii),Corr.2,10y,3rd&4th order
Case(ii),Corr.3,10y,3rd&4th order
Case(ii),Corr.4,10y,3rd&4th order
Case(iii),Corr.1,10y,3rd&4th order
Case(iii),Corr.2,10y,3rd&4th order
Case(iii),Corr.3,10y,3rd&4th order
Case(iii),Corr.4,10y,3rd&4th order
Case(iv),Corr.1,10y,3rd&4th order
Case(iv),Corr.2,10y,3rd&4th order
Case(iv),Corr.3,10y,3rd&4th order
Case(iv),Corr.4,10y,3rd&4th order
Differences
Moneyness
3rd
4th

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