"Banging the close": Fix orders and price manipulation

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February 1, 2016

Abstract

Recent years have seen several high-profile cases of manipulation of benchmark prices in financial markets, like the manipulation of the 4 pm fix for foreign exchange rates. A trading practice known as "banging the close" has featured in many of these cases. In a model of trading behavior and benchmark prices, I show that this practice occurs naturally as the solution of an optimal order execution problem. I use the model to study collusive behavior amongst traders, and extend the framework to allow for investors who rationally place fix orders. The results show the dangers of seemingly innocent information sharing between traders, and that a suggested change in the structure of currency benchmarks would likely be highly effective in alleviating disruptive trading practices.

*Comments and feedback are highly appreciated: *jo.a.saakvitne@bi.no*. I am particularly grateful to Bruno Gerard and Dagfinn Rime for extensive discussions. For helpful comments I would like to thank Jens Kvaerner, Namhee Matheson, Espen Skretting and seminar participants at the Finance Department, BI Norwegian Business School. All remaining errors are my own.

1 Introduction

In this paper I model a trading practice known as "banging the close". A trader who is banging the close is transacting a large amount of an asset exactly when a benchmark price is recorded (for example the closing price), in order to cause a large temporary price impact. The trader can benefit from the strategy by having some other position or larger scheme that benefits from the distorted benchmark price.

The practice of banging the close is seen in many markets. Recent examples include equities, spot oil and metal futures, where regulators have deemed the practice a form of price manipulation and fined market participants [1]. The perhaps best-known example of traders banging the close in recent years is the *foreign exchange (FX) fixing scandal*. According to regulators, traders at the some of world's largest banks colluded in banging benchmark rates on a large scale over a period of at least five years. A key structure in FX markets that gave traders an especially strong incentive to manipulate benchmark rates is the existence of so-called *fix orders*, an order placed by investors with traders securing execution at the yet unknown fix price.

In this paper I model traders receiving fix orders, and show that banging the close is an optimal order execution strategy. I also model why rational investors place fix orders even though they are aware of the trader's behavior. The model is then used to study the incentives for collusion between traders. I develop a simple game of information sharing where collusive front-running and benchmark manipulation occurs as Nash equilibrium trading strategies. An implication of this result is that seemingly innocent information sharing might facilitate a highly stable for of collusion that does not require any coordination mechanism or profit-sharing arrangement. The basic model is also extended to study a widening of the fix window, an adjustment to the FX benchmark rate setting process that has been proposed to hinder future manipulation. My results indicate that widening the fix window would be a highly effective measure.

The paper is structured as follows. I first give an account of the foreign exchange fixing scandal, and place my contribution in the context of the existing literature. The second part of the paper develops the model and its various extension. In the third part of the paper I discuss whether banging the close should really be classified as price manipulation. I also point to

[1] SEC fined the HFT firm Athena in 2012 for manipulating the close in Nasdaq stocks. CFTC fined Moore Capital Management in 2010 for banging the close in platinum and palladium markets. The firm Optiver was fined by the CFTC in 2012 for banging the close in oil markets, using a rapid-fire algorithm nicknamed *The Hammer*.

several avenues for further research opened up by this study.

The foreign exchange fixing scandal

Trading in foreign exchange markets averaged $5.3 trillion per day in April 2013 (Rime and Schrimpf, 2013), dwarfing in size any other financial market. FX markets never close, but trading activity is concentrated around European business hours for most currency pairs (Evans, 2014).

The most important benchmark price in FX markets is the London 4 pm fixing rate (the WM/Reuters fix). The benchmark is calculated as a sample average of prices over a 60-second interval at 4 pm London time. WMR Fixes are used for constructing indices comprising international securities, such as the MSCI indices and Barcalys Global Bond Index. They are also routinely used to compute the returns on portfolios that contain foreign currency denominated securities, such as country tracking funds and exchange traded funds (ETFs), as well as the value of foreign securities held in custodial accounts (Evans, 2014).

Trading volumes in major currency crosses spikes around the London 4 pm fix, as shown in figure 1. In a survey of fund managers and other investors, Financial Stability Board (2014) find it is common for investors to trade at the fix in order to reduce index tracking error:

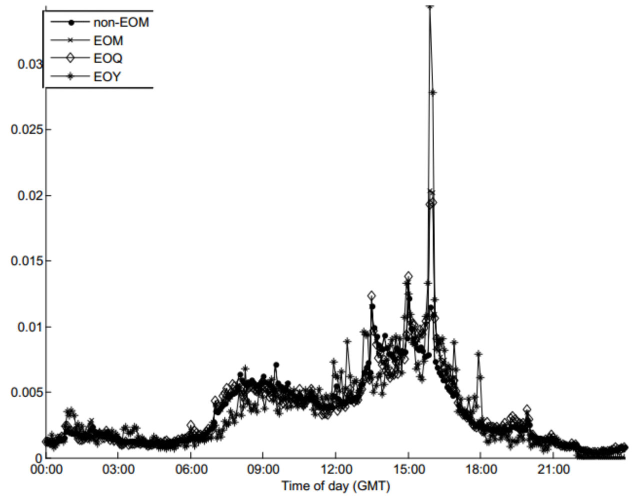
> Passively managed funds, including ETFs, are more likely *[than actively managed funds]* to use the WMR fix to minimize tracking error and meet mandate transfer requirements. Many asset managers noted that they had progressed to using the fix because of previous concerns about the non-transparency of bilateral pricing by custodian banks. *[...]* Most investment mandates are benchmarked against global indices that use the WMR 4pm London fixing for FX valuation and transaction purposes. As a result, there is a self-reinforcing dynamic whereby indices are benchmarked versus these fixes, investors tracking those indices seek to minimize their FX risk by transacting directly at those same fixes.

In the summer of 2013, news reports began to circulate that regulators were investigating manipulation of the London FX fixings.[2]

In November 2014, the United Kingdom's Financial Conduct Authority (FCA) imposed fines totaling $1.7 billion on five of the world's largest

[2] *Traders Said to Rig Currency Rates to Profit Off Clients*, Bloomberg, June 12 2013.

Figure 1: Intraday volumes in G10 currencies (Melvin and Prins, 2015)
The chart shows the percentage of trading occurring in 5-minute intervals, on the leading electronic brokerage platforms (EBS and Reuters) on average over the G-10 currencies.



banks[3] for failing to control business practices in their G10 spot foreign exchange trading operations. The FCA determined that the five banks had failed to manage risks around client confidentiality, conflict of interest, and trading conduct. The banks used confidential customer order information to collude with other banks to manipulate fixing rates for G10 currency rates and profit illegally at the expense of their customers and the market[4]. On the same day the United States Commodity Futures Trading Commission (CFTC) in coordination with the FCA imposed collective fines of $1.4 billion against the same five banks for attempted manipulation of, and for aiding and abetting other banks' attempts to manipulate global FX benchmark rates to benefit the positions of certain traders.

The CFTC found that currency traders at the five banks coordinated their trading with traders at other banks in order to manipulate the foreign exchange benchmark rates, including the 4 p.m. WM/Reuters rates.[5] Currency traders at the banks used private chatrooms to communicate and plan their attempts to manipulate the foreign exchange benchmark rates. In these chatrooms, traders at the banks disclosed confidential customer

[3]Citibank, HSBC, JPMorgan, RBS and UBS
[4]FCA press release 12 November 2014
[5]CFTC press release 12 November 2014

order information and trading positions, changed trading positions to accommodate the interests of the collective group, and agreed on trading strategies as part of an effort by the group to manipulate different foreign exchange benchmark rates. These chatrooms were often exclusive and invitation only, and were named for example *The Bandits' Club*, *The Mafia* and *The Cartel*.

On 20 May 2015, the five banks pleaded guilty to felony charges by the United States Department of Justice and agreed to pay fines totaling more than $5.7 billion. Four of the banks pleaded guilty to manipulation of the foreign markets. UBS also pleaded guilty to committing wire fraud and agreed to a $203 million fine. A sixth bank, Bank of America, while not found guilty, agreed to a fine of $204 million for unsafe practices in foreign markets.

Civil litigation from investors against the perpetrating banks are still ongoing. In 2015, nine banks negotiated some $2 billion in settlements[6]. The case, together with the high-profile manipulation scandal of LIBOR money market rates, have triggered a significant interest amongst market regulators in benchmark design. Several suggestions have been made for reforming the FX benchmark fixing, including widening the fixing window and establishing independent facilities for netting, order execution and order matching.

The existing literature

Market manipulation is as old as markets itself - soon after the Amsterdam Stock Exchange was founded at the beginning of the seventeenth century, brokers discovered that they could profitably manipulate stock prices. They would engage in a concentrated bout of selling to push down the price, in order to buy back at a profit (De La Vega, 1688).

The modern academic literature on market manipulation began with Hart (1977). Subsequent contributions to the early literature include Vila (1989), Allen and Gorton (1991), Allen and Gale (1992), Jarrow (1992), and Jarrow (1994). Much of this literature aims to define various forms of market manipulation within the paradigm of rational expectations. Later contributions have examined the role of rumors (Bommel, 2003), corners and squeezes (Allen et al., 2006) and manipulation specifically in the context of FX markets (Vitale, 2000). A well-developed strand of the literature related to the current paper are the studies considering manipulation of

[6] *7 banks make bid to exit forex price-fixing lawsuit*, Law360, 1 December 2015

closing prices in equity markets (Hillion and Suominen, 2004; Comerton-Forde et al., 2005; Comerton-Forde and Putniņš, 2011b,a).

A contribution of the current paper is to explicitly link market manipulation to optimal order execution strategies. The literature on order execution, dating back to Bertsimas and Lo (1998), has to my knowledge not emphasized this consideration previously. The starting point of the current model is the framework developed by Almgren and Chriss (2001) and subsequent papers (Almgren, 2003, 2012; Forsyth, 2011; Huberman and Stanzl, 2005; Kissell et al., 2003). These papers consider the problem of a trader with an "arrival price"-order. The trader solves for the optimal execution path, typically by solving a mean-variance maximization problem. The price impact of trading gives the trader incentives to smooth his trades over time, while the risk of adverse price movements gives incentives for a speedy execution of the order. Recently, Frei and Westray (2013) extend the framework to a value-weighted average price (VWAP) order, and characterize the optimal smoothing of trade execution. A closely related strand of the literature is concerned with optimal execution in a limit order book (Alfonsi et al., 2010; Bayraktar and Ludkovski, 2012; Cont and Kukanov, 2013; Løkka, 2013; Obizhaeva and Wang, 2013).

The optimal execution literature usually takes the price impact of trades as given, and the current paper follows this convention. Other branches of the market microstructure literature examines the *causes* of price impact. There are two dominant approaches to the cause of price impact: information frictions and inventory costs. Information frictions cause price impact because market participants run the risk of trading with a better informed counterparty (Copeland and Galai, 1983; Glosten and Milgrom, 1985; Kyle, 1985). Price impact of a trade can also occur when the counterparty bears an inventory cost from taking on the asset traded, for instance due to risk aversion (Ho and Stoll, 1981; Vayanos, 2001). While the formal model developed in this study is silent on the sources of price impact, these classical models should be kept in mind as a motivation for the structure.

2 Model

2.1 Structure and assumptions

The model follow standard assumptions from the optimal execution-literature. The basic model is presented in discrete time for simplicity of exposition. A continuous-time version of the model is derived in a later section for studying a widening the fixing window.

The basic model consider the decision problem of one trader in isolation. The trader has received an order to transact $\hat{x} \in \mathbb{R}$ units of an asset, at time $t = 0$. If the order is for selling assets, $\hat{x}$ will be a positive number. The order is not motivated by any private information: we are considering a "liquidity-motivated trade". The asset sale is to be completed at time $T \in \mathbb{N}$. The transaction of the trader at time t is denoted x_t, $t = 1, 2, \ldots, T$. A *trading strategy* is a tuple $\mathbf{x} \in \mathbb{R}^T$ satisfying assumption 1:

Assumption 1 (The trader must completely liquidate the order by time T)**.**

$$\hat{x} = \sum_{t=1}^{T} x_t$$

The fair value of the asset at time t is denoted v_t, and is considered a known quantity at time t. By fair price should be understood what the market price would have been at time t in the absence of any buying or selling pressure. In a dealer market, we can think of v_t as the price a competitive market maker would quote for a marginal trade, at a time when there is no order flow. The model abstracts from bid-ask spreads. One can, if one wish, interpret this as all transactions taking place at the bid side of the spread.

Assumption 2 (The fair price of the asset is a random walk with no drift)**.**

$$v_t = v_{t-1} + \sigma_v \epsilon_t, \;\; \epsilon_t \sim \mathcal{N}(0, 1) \; \textit{for all } t$$

The optimal execution strategy of the trader in fact only requires the weaker condition that the fair asset price is a martingale. The parametric specification for the price process is made in order to compute variances of various payoffs. Assumption 2 is chosen for transparency and tractability. For many assets, for instance stocks and bonds, a long-term value process should include a drift term. However, the short time frame we are considering, usually a day or less, means that the random walk with no drift is a good approximation. In recent years, progress has been made to solve optimal execution problems for other processes (Gatheral and Schied, 2011; Forsyth et al., 2012), and a topic for future research could be to implement these solutions for the fix order.

The key ingredient to optimal execution models is the price impact of trades, inserting a wedge between the fair price of the asset and the market price. The underlying assumption made here is that there is sufficient liquidity provisions made by agents outside the model so that all orders can be filled, conditional on an impact to market prices.

The price impact can be thought of as a compensation for potential information asymmetries (Copeland and Galai, 1983; Glosten and Milgrom, 1985; Kyle, 1985), or inventory costs of liquidity providers (Ho and Stoll, 1981; Vayanos, 2001). The impact can conceivably be both temporary and permanent, and a variety of functional forms have been studied (Gatheral, 2010).

Since my aim with the model is to examine fix orders in the simplest, most transparent manner, I disregard any permanent price impact from trading. For further simplicity, I consider the temporary price impact to be a *linear function* of transaction size. The fixed and positive coefficient of temporary price impact is denoted by η.

Assumption 3 (Temporary price impact is a linear function of trade size)**.**

$$p_t = -\eta x_t + v_t$$

Associated with each trading strategy $\mathbf{x}$ is a payoff for the trader, π. The payoff is a random variable, as it depends on the realized path of the fair price as well as the trading strategy.

The trader is risk-neutral. The case of a risk-averse trader with mean-variance preference is considered in an extension later in the paper. There has been recent progress in modeling optimal execution for more general preference structures (Schied and Schöneborn, 2009), and there exists a potential for future research in applying these results to the fix order. Payoffs are not discounted, due to the short time horizons involved. Denote the utility of the trader from payoff π as $U_T(\pi)$.

Assumption 4 (Trader is risk-neutral)**.**

$$U_T(\pi) = \mathbb{E}\pi$$

2.2 Optimal execution - the basic model

The main concern of this paper is the optimal execution of fix orders. It will however be useful to first derive the optimal execution of another order type within the same model framework, namely the arrival price-order. This is useful for three reasons. First, the execution of arrival price-orders is the most well studied execution problem in the literature (see e.g. Almgren and Chriss (2001)), and as such form an important baseline and introduction to the model. Second, the execution of an arrival price-order within the current modeling framework corresponds to minimizing trading costs. This makes the arrival-price a useful reference point and, as it will turn out, a stark contrast to the fix order. Third, the derivation of the optimal strategy for a fix order builds directly on the strategy for the arrival price order, which makes it natural to develop the arrival price-order first.

Execution of arrival price-orders

The risk-neutrality of the trader yields a particularly simple optimal trading strategy for the arrival-price order: to spread trading evenly out over time and thereby minimize trading costs.

The first step is to specify the profit/loss of the trader. This quantity, also known as "slippage" in the order execution literature, is the difference between the actual execution price and the price at time 0. The model allows for the trader to receive a fixed fee b_i from the investor, an object that will come into play when we later consider the decision problem of the investor.

$$\mathbb{E}\pi = \mathbb{E}\left[\sum_{t=1}^{T} p_t x_t - p_0 \hat{x} + b_a\right]$$

Insert the price impact from A3, and use that by A2, the asset price is a martingale:

$$\mathbb{E}\pi = p_0 \sum_{t=1}^{T} x_t + \eta \sum_{t=1}^{T} x_t^2 - p_0 \hat{x} + b_a$$

Using that trades must sum to order size (A1), we can reduce the trader's expected profit to simply the summed price impact of trading:

$$\mathbb{E}\pi = \eta \sum_{t=1}^{T} x_t^2 + b_a$$

The risk-neutral trader choose the trading strategy which maximize his expected profit:

$$\max_{\mathbf{x}} \left(\eta \sum_{t=1}^{T} x_t^2 + b_a\right) \text{ s.t. } \left(\sum_{t=1}^{T} x_t = \hat{X}\right) \tag{1}$$

Proposition 1. the optimal execution strategy of an arrival price-order is to spread the trade evenly out over all periods:

$$x_i = \frac{1}{T}\hat{x}$$
$$i = 1, 2, \ldots, T$$

Proof. Let γ denote the Langrange multiplier:

$$\mathcal{L} = \eta \sum x_t^2 - b_a - \gamma \left(\sum x_t - \hat{x} \right)$$

We get T first-order conditions (FOCs) of the form

$$x_t = \frac{1}{2\eta}\gamma$$

Sum all the FOCs, and use that $\sum x = \hat{x}$ by the constraint:

$$\sum x_t = \hat{x} = \sum^T \frac{1}{\eta}\gamma = \frac{T}{2\eta}\gamma$$

Insert that $\gamma = 2\eta x_i$ for each i:

$$x_i = \frac{1}{T}\hat{x}$$

$$i = 1, 2, \ldots, T$$

□

Proposition 1 summarize the intuitive result that the optimal execution strategy of the risk neutral trader is to spread the trade evenly out over all periods.

Execution of fix price-orders

The key feature of the fix order is that the trader guarantees that the customer receives the fix price for the assets. The trader makes a profit or a loss depending on how actual sales income (for the case of a sell order) differs from the fix price:

$$\mathbb{E}\pi = \sum_{t=1}^{T} \mathbb{E}\left[p_t\right] x_t - \mathbb{E}[p_T]\hat{x} + b_f \tag{2}$$

Where as before b_f is a fee for the order execution paid by the investor.

Inserting the price impact function (A3) and using the martingale property of the price process (A2), we can rewrite the maximization of expected profit as:

$$\max_{\mathbf{x}} \left(\eta \sum_{t=1}^{T} x_t^2 + \eta x_T \sum_{t=1}^{T} x_t + b_f \right) \text{ s.t. } \left(\sum_{t=1}^{T} x_t = \hat{x} \right) \tag{3}$$

Equation (3) is a dynamic optimization problem. However, by proposition 2 we can rewrite it to essentially a two-period maximization problem. The proposition says that the trader's optimal strategy is to execute the same amount for all periods except the fixing period.

Proposition 2. The T-tuple $\mathbf{x}$ which maximize equation (3) has $x_i = x_j$ for all $i, j \neq T$

Proof. For any fixed x_T, the problem of finding the remaining x_i's in (2) is

$$\max_{\{x_i\}_{i=1}^{T-1}} \left(\eta \sum_{i=1}^{T-1} x_i^2 - \eta x_T \sum_{i=1}^{T-1} x_i \right) \text{ s.t. } \left(\sum_{i=1}^{T-1} x_i = (\hat{x} - x_t) \equiv \bar{x} \right)$$

Define $\hat{x} - x_T \equiv \bar{x}$, $T' = T - 1$ and $\mathbf{x'} \equiv \{x_t\}_{t=1}^{T'}$. Using this, we can rewrite the above as

$$\max_{\mathbf{x'}} \left(\eta \sum_{t=1}^{T'} x_t' - x_T \bar{x}) \right) \text{ s.t. } \left(\sum_{t=1}^{T'} x_t' = \bar{x} \right)$$

This is now the same problem as (1), apart from the constant term $x_T \bar{x}$ which is irrelevant for the optimal strategy. Hence the proposition follows immediately from applying proposition 1. □

The problem of finding the optimal trade sequence has now essentially been reduced to the two-stage allocation problem of deciding how much to sell in the $(T-1)$ first periods, versus how much to sell in the fixing period. The trade-off here is intuitive; the transaction in the fixing period is in itself unprofitable for the trader, but since it pushes down the fixing price, trading large volumes at the fix indirectly generates profit on all earlier trades.

The second-stage constrained optimization problem is easily solved with the help of a Lagrange multiplier. Denote by x_i the trade size in each period except the fixing period, and use that by proposition 2, we have that $\sum_{i=1}^{T-1} x_i = (T-1)x_i$. The Langrangian becomes

$$\mathcal{L} = \eta(T-1)x_i x_T - \eta(T-1)x_i^2 - \gamma(x_i(T-1) + x_T - \hat{x})$$

Where γ denotes the Langrange multiplier. The three first order conditions are:

$$x_T - 2x_i = \gamma$$

$$\lambda = x_i(T-1)$$

$$\hat{x} = (T-1)x_i + x_T$$

Note that the optimal execution path of the trader does not depend on the impact coefficient η. This is due to the linearity of the price impact.

Solving this system of equations for x_i and x_T gives us the optimal execution strategy of the trader, summarized in proposition 3.

Proposition 3. For all periods except the fixing period, the trader sells a constant amount proportional to the total number of periods:

$$x_{t \neq T} = \frac{1}{2T}\hat{x}$$

While in the fixing period, the trader sells close to half of the total order:

$$x_T = \frac{T+1}{2T}\hat{x}$$

Proof. In the text above. □

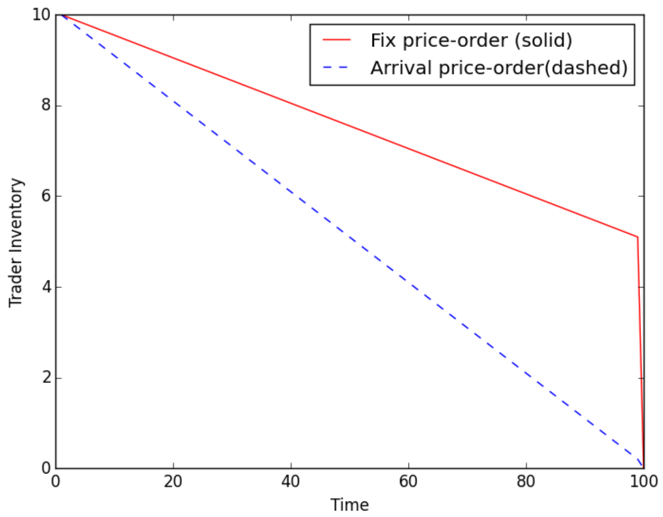
If we divide the day into an increasing number of trading periods, a straightforward application of L'Hôpital's rule shows that the trader executes close to half of the volume the final period:

$$\lim_{T \to \infty} x_T = \frac{1}{2}\hat{x}$$

If we let $X_0 = \hat{x}$, we can write the inventory of the trader at the end of period t as $X_t = X_{t-1} - x_t$. Figure 2 below compares the optimal time-path of the trader's inventory for an initial price-order and a fixing price-order. The solid line shows how the trader sells half his inventory gradually during the day, and then dumps his remaining inventory on the market exactly when fix price is set. Since the price impact of this final trade pushes down the benchmark fix price severely, the trader makes an expected profit from all the preceding transactions. The optimization problem of the trader is essentially to decide how much of the initial order to transact through the day, and how much to use for pushing the fix price down. The execution strategy of the arrival price-order is drawn as the dotted line in the same figure.

In the model of this section, the price is benchmarked to a fix at the end of the day. However, it would not matter if the fix where instead at any other point in time *after* the trader has received the order. The trader will trade equally over all periods, except the fixing period.

Figure 2: Optimal inventory - the trader is "banging the close"



Fix orders and market manipulation

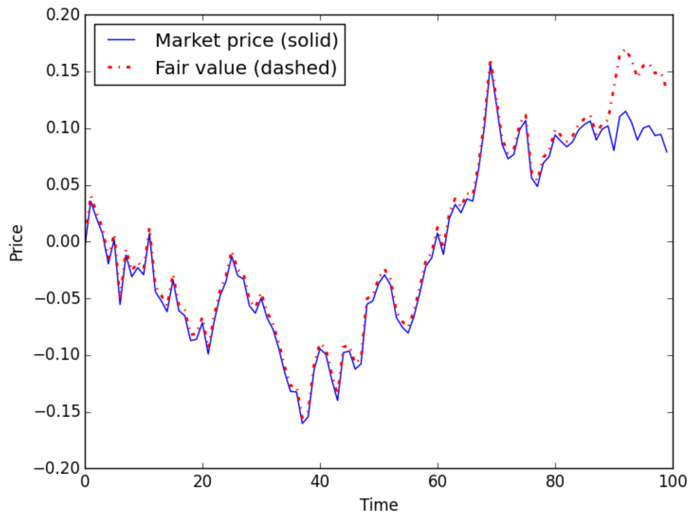
We have derived the optimal execution strategy of fix orders. It is instructive to look at figure 3 to see why real-world traders call this kind of strategy "banging the fix". This plot show a simulation run of the model from section 2.2 where the fix window is the last 10 per cent of the day (in real-world FX markets it is around 0.2 per cent). For the majority of the day, the fair value and the market price are almost exactly overlaying each other. At the fix, however, there is a significant gap between the two. If we think of the market price as a signal for the fair value of the asset, the signal is clearly distorted during the fix. We will return to this issue in the discussion on disruptive trading practices and price manipulation.

Risk averse traders

An assumption maintained throughout the paper is that traders are risk neutral (A4). We now consider instead the execution strategy of a trader with preferences that can be represented by a mean-variance type utility function. The main result of the analysis is that the risk averse trader has two coinciding motives for trading a large amount at the fix: to perform risk-management and to cause a price impact.

To intuitively see the strategy of the risk averse trader with a fix order, it is again instructive to first consider how he executes an arrival price-order. It that case, the trader "front-loads" the execution by selling more in ear-

Figure 3: Sample price evolution - the trader is "banging the close"



lier periods, and less later in later periods (Almgren and Chriss, 2001). In the extreme case, when the risk aversion is very high (or the price impact negligible), the trader will execute the entire order at once, and thus have no risk of slippage from his price benchmark.

Now consider the risk averse trader with a fix order. As with the arrival price-order, the trader will execute a larger part of the volume close to the time of price benchmarking. In other words, this trader will make an even larger price impact than the risk neutral trader.

The profit of trader with a sell order benchmarked to the fix can be written as

$$\pi = \sum_{i=1}^{T}(p_i - p_T)x_i$$

The expected profit is already computed in the previous section (equation (3):

$$\mathbb{E}\pi = \eta(-\sum_{i=1}^{T} x_i^2 + x_T \sum_{i=1}^{T} x_i)$$

To compute the variance, we first insert the price impact function $p_i = v_i - \eta x_i$ (A3) into the expression for trader profit:

$$\pi = \sum_{i=1}^{T} x_i(v_i - v_T) + \eta \sum_{i=1}^{T}(x_T x_i - x_i^2)$$

Now we use that the price is a random walk (A2), and hence the price difference between two periods is simply the sum of the disturbances:

$$v_i - v_T = -\sum_{j=i+1}^{T} \sigma\epsilon_j$$

Inserting this in the expression for π:

$$\pi = -\sum_{i=1}^{T} x_i \sum_{j=i+1}^{T} \sigma\epsilon_j + \eta \sum_{i=1}^{T} (x_T x_i - x_i^2)$$

Only the first term in the above expression for π is stochastic. Hence, the variance is given by

$$Var(\pi) = Var\left(\sum_{i=1}^{T} x_i \sum_{j=i+1}^{T} \sigma\epsilon_j\right)$$

$$= \sigma^2 Var(x_1(\epsilon_2 + \epsilon_3 + \cdots + \epsilon_T) + x_2(\epsilon_3 + \cdots + \epsilon_T) + \cdots + x_{T-1}\epsilon_T)$$

$$= \sigma^2 Var(\epsilon_2 x_1 + \epsilon_3(x_1 + x_2) + \cdots + \epsilon_T(x_1 + \cdots + x_{T-1}))$$

The second equality follow from the independence of price changes, recall that $\epsilon_i \sim \mathcal{N}(0,1)$ for all i. Hence, the profit variance is proportional to the cumulative sum of squared trade volume:

$$Var(\pi) = \sigma^2 \sum_{i=1}^{T-1} \left(\sum_{j=1}^{i} x_j\right)^2 \tag{4}$$

To illustrate the effect of risk aversion, I solve the maximization problem (5) numerically.

$$\max_{\{x_t\}} \left[\mathbb{E}\pi - \gamma Var(\pi)\right] \tag{5}$$

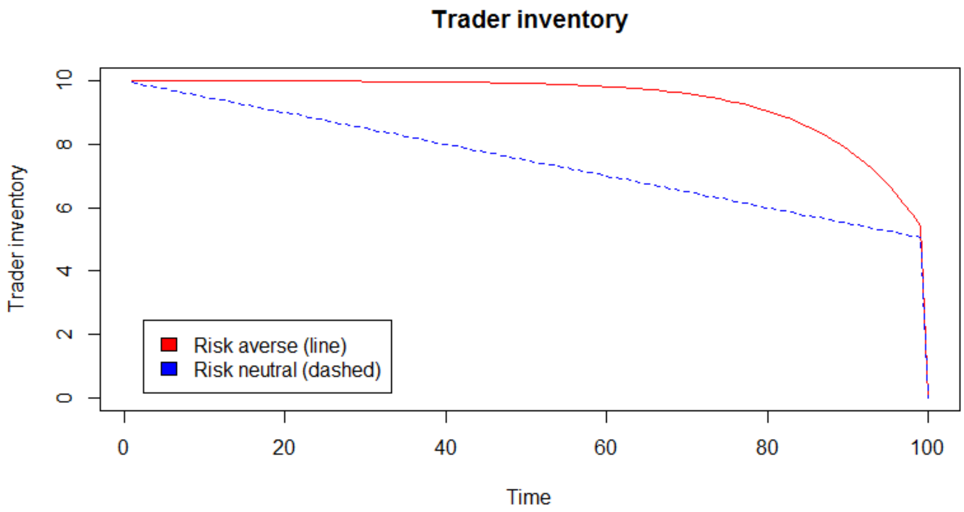
Table 1 shows the parameter values used. The resulting trade sequence is shown in figure 4.

The numerical example plotted in figure 4 illustrates the two ways in which risk aversion qualitatively changes the optimal execution strategy. First, the risk averse trader executes a larger share of the total order at the time of the fix. This is because the risk-averse trader has a lower valuation of the payoff from executing small trades before the fix. Second, the risk averse trader no longer smooths the trading linearly in the periods before the fix, but instead he trades very little early in the day and gradually ramps up trading as the fix is getting closer. Traders have both a "legitimate" inventory management motive for trading, as well as "manipulative" motive of

Table 1: Parameter values

Asset holdings ($\hat{x}$)	10
Periods per day (T)	100
Annual volatility	0.2
Per-period vol. (σ^2)	0.00105
Risk aversion (λ)	2.0
Price impact (η)	0.3

Figure 4: Optimal inventory for risk-averse trader



causing a price impact. Indeed, FX traders commenting on the fixing scandal have pointed to risk management as a cause for the observed trading patterns. While these two motives may be hard to separate in real-world data, comparing the dashed and drawn line in figure 4 shows clearly the additional effect of risk aversion in this model.

2.3 Rational investors

The basic model derived in section 2.2 can be extended to study if and why rational investors place fix orders. The main result is proposition 4, which spells out a simple condition for fix orders to be optimal. The extension of the model is built on a structure summarized by three additional assumptions.

The first additional assumption is that the market for trading services is competitive, so that traders earn zero expected profit. w_i denotes the payment from the trader to the investor.

Assumption 5 (Zero trader profit)**.**

$$\mathbb{E}(\pi - w_i) = 0$$

The second assumption specifies the structure of the payment w between the trader and the investor. The trader is selling an amount $\hat{x} > 0$ of assets on behalf of the investor (the same equations hold for a buy order, where $\hat{x} < 0$) , either as a fix order or as an arrival price-order.

The trader pays the investor the sales income $p_i\hat{x}$, where p_i is either the fix price p_T or the arrival price p_0. The investor also pays the trader a lump-sum compensation for transaction services (a "fee"). The role of the fee is to ensure that the zero-profit condition of traders hold. It will turn out that for the fix order, the fee is negative - in other words are the traders willing to pay investors for fix orders.

The structure outlined above corresponds to a linear pricing scheme for order executions, contracts where the trader pays the investor a linear function $w(\hat{x})$. Each of the two order types (arrival price- and fix price-orders) corresponds to a different linear function, indexed by $i = \{\text{arrival}, \text{fix}\}$.

Assumption 6 (Linear pricing of orders)**.**

$$w_i = p_i\hat{x} - b_i$$

$$p_i = \begin{cases} p_0 & \text{if arrival price-order} \\ p_T & \text{if fix price-order} \end{cases}$$

The third assumption pins down the preferences of the investor. The investor cares about the price he gets from selling his assets, *relative to the benchmark price*. The benchmark price is the fix price p_T . This assumption captures the idea of the investor as an asset manager who is incentivized by deviation from an index, as discussed earlier in the paper.

The preferences of the investor can be represented by a mean-variance type objective function: he wants to get the highest price possible for the assets, but at the same time dislikes the risk of transacting at prices far worse than the benchmark price.

Assumption 7 (Investor has mean-variance preferences)**.**

$$U_I(w_i) = \mathbb{E}\left[w_i - p_T\right] - \lambda Var(w_i - p_T)$$

We now have the necessary ingredients in place to characterize the decision problem of the investor: whether to place an arrival price-order or a fix price-order. He takes into account the execution strategy of the trader, and choose the order type that gives him the highest utility:

$$\max_{i \in \{fix,\ arrival\}} \{U_I(w_i)\} \tag{6}$$

Proposition 4. The investor will choose a fix order if the price impact is low, risk aversion or asset variance is large, or it is a long time till the fix:

$$\eta < \frac{4T^2}{T-1}\lambda\sigma^2$$

.

Proof. It follows from equation (6) that the investor chooses a fix order when $U_{fix} > U_{arrival}$. The proposition will follow from inserting the model structure, earlier results and the fees b_i into this condition. First, we determine the fees.

The fee is determined by the zero-profit condition of traders. Trader profit is found by inserting the optimal execution strategy into the profit functions. In the case of an arrival price order, this means inserting $x_t = \frac{\hat{x}}{T}$ into equation (1):

$$\mathbb{E}\pi_{arrival} = \eta \sum^{T} \frac{\hat{x}^2}{T^2} - b_a = \eta \hat{x}^2 \frac{1}{T} - b_a$$

By the zero-profit condition A5:

$$b_a = \frac{\eta \hat{x}^2}{T} \tag{7}$$

The intuition behind equation 7 is that execution costs are high when price impact of trading is high, or when the total order size is large. Execution costs are decreasing with a longer time horizon of the order (T), since the order can be spread over more trading periods.

We find the fee for the fix order by the same method. First insert the optimal execution strategy (3) back into (3) to find the profit of the trader:

$$\mathbb{E}\pi_{fix} = -\eta \sum_{t=1}^{T-1} x_t^2 - \eta x_T \sum_{t=1}^{T-1} x_t + b_i$$

Then use the zero-profit condition of traders and solve for b_f, to get

$$b_f = \frac{\hat{x}^2 \eta (1-T)}{4T} \tag{8}$$

Notice that since $T > 1$, the trader's fee is a negative number. This mean that traders in a competitive market will actually be willing to pay for fix orders, because traders can profit from "banging the fix".

We now consider the investor's utility from the two different orders. First, the fix order:

$$U_I(w_{fix}) = \mathbb{E}(w_{fix} - \hat{x}p_T) - \lambda var(w_{fix} - p_T)$$

where the income of the investor w_{fix} is given by assumption A6 and equation (8):

$$\mathbb{E}(w_{fix}) = p_T\hat{x} - b_{fix} = p_T\hat{x} - \hat{x}^2\eta\frac{1-T}{4T}$$

Since the variance of the payoff from the fix order is zero from the perspective of the investor ($var(w_{fix} - p_T\hat{x}) = 0$), we have that the investor's utility from placing the fix order is

$$U_I(w_{fix}) = p_T\hat{x} - b_{fix} = p_T\hat{x} - \hat{x}^2\eta\frac{1-T}{4T}$$

Then, we determine the utility from placing an arrival price-order. The investor's expected income from the arrival price order is given by A6 and equation (7):

$$\mathbb{E}(w_{arrival}) = \hat{x}\mathbb{E}(p_0 - p_T) - \hat{x}\eta\frac{1}{T}$$

The variance of the income can be calculated by using A2:

$$var(w_{arrival} - p_T\hat{x}) = var(p_0\hat{x} - p_T\hat{x}) = \hat{x}^2T\sigma^2$$

And we therefore find that $U_I(w_{arrival})$ is given by:

$$U_I(w_{arrival}) = \hat{x}\eta\left(\frac{T+1}{2T} - \frac{1}{T} - \frac{1}{\eta}\lambda T\sigma^2\right)$$

We can now insert the above into $U_I(w_{fix}) > U_I(w_{arrival})$, and simplify to get the condition

$$\eta < \frac{4T^2}{T-1}\lambda\sigma^2$$

□

The investor choosing between an arrival price order and a fix order face the following trade-off: If he place an arrival price order, the expected sales income he gets from his assets is higher than for the fix order, since he knows that the trader will bang the close with the fix order, thereby causing a large price impact. On the other hand, the risk-averse investor dislikes the uncertainty of transacting at a price far from his benchmark . This last consideration benefits the fix order, and the condition in proposition 4 tells us when this second consideration dominates the first.

2.4 Collusion

We now turn to the collusion incentives of traders with fix order. These incentives are modeled by adding one more trader to the basic model described in section 2.2. To keep things simple, the time line has only two trading rounds ($t = 1, 2$). I also disregard both arrival price-orders and the investor's decision problem, meaning that all orders are fix orders and that there are no fees paid to investors.

Let there be two traders, with fix orders $\hat{x}$ and $\hat{y}$. An order to sell is represented by a positive number. There are two periods, so the problem of trader x is

$$\max_{x_1, x_2} \{\mathbb{E}\pi^x\} \text{ s.t. } \hat{x} = x_1 + x_2 \tag{9}$$

$$\pi^x = p_1 x_1 + p_2 x_2 - p_2 \hat{x}$$

The problem of trader y is analogous.

Since price uncertainty does not affect the decisions of the risk neutral investor, as shown in section 2.2, I keep notation simple by normalizing the starting price to 0 and omitting the stochastic term of the price process:

$$p_i = -\eta(x_i + y_i) \tag{10}$$

We will study the decision of the traders to collude or not, taken at a time *before* either trader has knowledge of their own fix orders. We therefore let $\hat{x}$ and $\hat{y}$ be random variables, with mean zero and covariances known to both traders:

Assumption 8 (Order flow covariances)**.**

$$\mathbb{E}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\mathbb{E}\left[\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}^T\right] = \begin{pmatrix}\sigma_x^2 & \sigma_{xy}\\ \sigma_{xy} & \sigma_y^2\end{pmatrix}$$

We can distinguish between two different forms of collusion. The first form we can label *information sharing*:

Traders engaging in information sharing reveal their orders to each other before execution starts. After revealing their order sizes, each trader executes his own order optimally, taking into account the optimal order execution of the other trader. In game theoretic terms, the Nash-equilibrium prevails. This form of collusion has the advantage (from the perspective of the traders involved) that it disposes of the need for additional coordination and monetary transfers between the colluding agents. An intuitive picture of what is going on is that the involved traders meet in a chat room to discuss their order flow that day, a phenomena known in for example FX markets.

The second form of collusion we consider is a full-fledged *cartel*. Traders in a cartel pool their orders and execute as one, larger trader.

Information sharing

Collusion by information sharing is modeled as follows:

At time 0, trader x and trader y can choose to share information. The traders make this choice before their respective orders $\hat{x}$ and $\hat{y}$ are known. After the choice has been made to share information, realizations of the orders are drawn.

Each trader executes his order optimally, taking into account the optimal order execution of the other trader. The pair of optimal execution strategies therefore corresponds to the Nash equilibrium in an imagined game between the two traders (Nash, 1950).

To solve for the Nash equilibrium, consider the profit maximizing strategy of trader x. We find this by inserting equation (10) into (9) and simplify into an expression in x_1 and y_1:

$$\pi^x = p_1 x_1 + p_2(x_2 - \hat{x})$$

$$= x_1(x_1 + y_1 - x_2 - y_2)[-\eta]$$
$$= x_1(2x_1 + 2y_1 - \hat{x} - \hat{y})[-\eta]$$

The above profit function is convex in x_1. Hence, by the first order condition for a maximum:

$$\frac{\partial \pi^x(x^*)}{\partial x_1} = 0$$

$$\implies x_1^* = \frac{1}{4}(\hat{x} + \hat{y}) - \frac{1}{2}y_1 \tag{11}$$

This is the best response-function of trader x. To find the Nash-equilibrium, insert for y_1^* in the response function. By symmetry we immediately get $y_1^* = \frac{1}{4}(\hat{x} + \hat{y}) - \frac{1}{2}x_1$. Thus,

$$x_1^* = \frac{1}{6}(\hat{x} + \hat{y}) \tag{12}$$

$$x_2^* = \frac{5}{6}\hat{x} - \frac{1}{6}\hat{y}$$

The strategy for trader y is analogous.

There are several interesting features of this solution. First: even if the trader has no order ($\hat{y} = 0$), the optimal amount executed by trader x at the fix is still higher than in the case of no information sharing ($\frac{5}{6}\hat{x}$ vs. $\frac{3}{4}\hat{x}$). A second feature is that trader y will trade even if he has no order ($y_1 \neq 0$ even though $\hat{y} = 0$). Third, traders might now use a combination of buy and sell orders.

The reason for these features is that even if he has no order, trader y will "trade against" trader x by setting $y_1 = -y_2$, in a sense "free-riding" on the order of trader x by profiting from the price impact of by trader x's execution. If the difference in order size is sufficiently large, it will even be the case that a trader use buy orders as part of a sell order-program (for example if $\hat{x} = 12, \hat{y} = 2$), in order to profit from the large price impact caused by the other trader.

A fourth notable feature of the information sharing-model of collusion is that the profit function takes the same value for the two traders, independent on the individual trader's order size:

$$\pi^x = \frac{\eta}{18}(\hat{x} + \hat{y})^2 = \pi^y$$

In other words, *knowing* about the order is equally valuable as actually *having* the order.

Do traders benefit from collusion by information sharing? We can answer this question by investigating if expected profit is higher with information sharing, than when the trader operates individually. For trader x in a case with no information sharing, profit is

$$\pi^x_{individual} = \frac{\eta}{8}\hat{x}^2$$

The risk neutral traders have preferences only over expected profit. The traders of the model will therefore prefer information sharing if expected profits are higher:

$$\mathbb{E}\pi_{share} > \mathbb{E}\pi_{individual}$$

Using A8, this condition can be rewritten into

$$corr(\hat{x}, \hat{y}) > \frac{\frac{5}{8}\sigma_x^2 - \frac{4}{8}\sigma_y^2}{\sigma_x \sigma_y}$$

The corresponding condition for trader y is:

$$corr(\hat{x}, \hat{y}) > \frac{\frac{5}{8}\sigma_y^2 - \frac{4}{8}\sigma_x^2}{\sigma_x \sigma_y}$$

The interpretation of these conditions is the following: both traders will benefit from sharing information if their orders are sufficiently correlated. For the special case where the traders are equal ($\sigma_x = \sigma_y$), information sharing is profitable if

$$corr(x, y) > \frac{1}{8}$$

If the traders are not equal ($\sigma_x \neq \sigma_y$), the correlation needs to be higher than if they are equal. The reason is the "free-riding behavior" discussed earlier. Since one of the conditions above will always fail to hold if correlation is negative, information sharing is infeasible between traders with negatively correlated order flows.

How realistic is the requirement for correlated order flows? In the case of FX markets, it might actually be quite realistic. A case in point is Melvin and Prins (2015), who show that FX fix price movements are associated with preceding stock index changes, indicating that equity rebalancing flows make up a significant proportion of fix order volume. FX dealers often get a "one-sided" order flow preceding the fix, which in model terms translates into correlated orders.

The cartel

I now consider the profitability of a full-fledged cartel between a group of traders. If trader x pools his order with trader y and they act as one, trader x can achieve expected profit $\mathbb{E}\pi^x_{cartel}$:

$$\mathbb{E}\pi^x_{cartel} = \alpha \frac{1}{8}(\hat{x} + \hat{y})^2 \eta$$

$\alpha \in (0, 1)$ is an exogenous "sharing rule", specifying the share of joint profit that goes to trader x. Similarly, trader y's profit from the cartel is

$$\mathbb{E}\pi^y_{cartel} = (1 - \alpha)\frac{1}{8}(\hat{x} + \hat{y})^2 \eta$$

Expand $\mathbb{E}\pi^i_{cartel} > \mathbb{E}\pi^i_{individual}$ for $i = x, y$ to find the condition for when the cartel is preferred over no collusion:

$$\frac{\frac{\sigma_x}{\sigma_y}}{\frac{\sigma_x}{\sigma_y} + \frac{\sigma_y}{\sigma_x} + 2\sigma_{xy}} < \alpha < 1 - \frac{\frac{\sigma_y}{\sigma_x}}{\frac{\sigma_x}{\sigma_y} + \frac{\sigma_y}{\sigma_x} + 2\sigma_{xy}}$$

$$\Leftrightarrow$$

$$\frac{\rho^2}{1+\rho^2+2\rho r_{xy}} < \alpha < \frac{\rho^2+2\rho\sigma_{xy}}{1+\rho^2+2\rho r_{xy}} \tag{13}$$

where $r_{xy} := \sigma_{xy}\sigma_x^{-1}\sigma_y^{-1}$ is the correlation coefficient between orders, and ρ is the ratio between the standard deviation of order flows:

$$\rho := \frac{\sigma_x}{\sigma_y}$$

The condition in equation (13) show that traders can benefit from pooling their orders as long as their order flows are not negatively correlated. To see why, consider the three possible cases for correlation separately.
First, consider the case when correlation between order flows is zero ($r_{xy} = 0$). Condition (13), together with the fact that variance is always a positive number, implies a unique α between zero and one:

$$\frac{\rho^2}{1+\rho^2} = \alpha \in (0,1)$$

Hence, if correlation between orders is zero, there always exists a unique sharing rule α such that the cartel is feasible.

If, on the other hand, correlation is positive, the sharing rule is no longer unique. But there will still always be an interval for α where collusion is profitable for both traders. The third possible case is that the correlation between orders is negative. Then, there will not exist an α such that collusion is profitable. To see why these two last statements are true, rewrite condition (13) into

$$\rho^2 < K\alpha < \rho^2 + 2\rho r_{xy}$$

Where $K > 1$ is a constant and $r_{xy} \in [-1, 1]$.

We have showed that if correlation between orders are not negative, there exist a sharing rule such that the cartel (traders pooling orders and acting as one) is profitable for both traders. It should be noted that the cartel model is rather limited, in the sense that it lack features such as credibility of commitment and mechanisms for profit sharing. Building a richer model of these cartels is not within the scope of the present paper, but seems like a promising venue for future research.

Both varieties of collusion puts some conditions on the correlation of orders. These requirements are quite intuitive. For information sharing to be feasible, the traders involved must expect to have similar interests regarding whether the fix price should be high or low. This is captured by a requirement for positively correlated orders.

If, on the other hand, the traders join up in a cartel, they can jointly cause a larger price impact than they can individually (since optimal price impact is quadratic in order size). Therefore, a cartel is profitable even if orders are uncorrelated. The cartel could however also require some form of profit sharing-mechanism and a credible commitment, features which are not modeled here.

2.5 Extending the fix

A suggestion that has been made to remedy the manipulation of fixing prices is to widen the time window where the fix price is determined (see for example Financial Stability Board (2014)). In this section I derive a continuous time-formulation of the model in section 2.2 and use it to study how lengthening the fix window affects the optimal execution of fix order and the associated price distortion. The conclusion is that a lengthening would be a highly effective measure in reducing price distortion.

The state variable of the trader is his inventory. Trader inventory at time t is denoted $x(t)$, and hence the liquidation rate is the time derivative of inventory ($\dot{x}$).

The assumption of total order liquidation (A1) becomes

$$x(0) = x_0$$

$$x(T) = 0$$

For a buy order $x_0 < 0$, while for a sell order $x_0 > 0$.

The price impact from trading is still linear in trade size, so assumptions A2 and A3 becomes:

$$p(t) = \eta\dot{x} + w(t)$$

Where $w(t)$ is a martingale process.

Trader profit is always (*cash in*) - (*cash out*). I continue to write in terms of a sell order, while of course the equations are equally valid for buy orders. For a sell order, cash out is (*order size*)*(*average price in fixing period*), while cash in is simply integrating up the product (*volume)*(price*):

$$\pi = -\int_0^T p(t)\dot{x}dt - x_0\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)dt$$

In expectations the w-terms cancel out since $\mathbb{E}(w) = w(0)$. Therefore, the expression for expected profit is:

$$\mathbb{E}\pi = -\int_0^T \eta \dot{x}^2 dt - x_0 \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \eta \dot{x} dt \tag{14}$$

The problem of the trader is to find $x(t)$ that maximize expected profit $\mathbb{E}\pi$. I consider the case where the fix is at the end of the day ($t_2 = T$).

The solution method follows these three steps:

1. Take as given an allocation to the fix window, and solve for optimal path during the day
2. Do the same for fix window
3. Solve the two-period problem of allocation to the fix window versus rest of the day

Optimal path of pre-fix period

The optimal sales intensity turns out to be a constant, meaning that the trader's inventory is a linear function of time.

The problem of the trader is

$$\max \left\{ -\int_0^{t_1} \eta \dot{x}^2 dt \right\}$$

subject to

$$x(0) = x_0$$
$$x(t_1) = x_1$$

where x_1 is the amount allocated to the fix window and x_0 is the total order.

The above is a standard optimal control problem with the inventory $x(t)$ as the state variable, and control u given by:

$$u(t) = \dot{x}$$

I solve by applying Pontryagin's maximum principle (see for example Seierstad and Sydsaeter (1986)). First, form the Hamiltonian:

$$H = q_0 \eta [u(t)]^2 + q(t) u(t)$$

Since $\dot{q} = H'_x = 0$, the adjoint function $q(t)$ is a constant:

$$q(t) = C_1$$

Insert this back into the Hamiltonian and use that the optimal control u^* maximize H. The FOC is

$$2uq_0\eta + C_1 = 0$$

Solve for the optimal control, and use that since $q_0 \in \{0, 1\}$, $q_0 = 1$ for the problem to make sense:

$$u^* = -\frac{C_1}{2q_0\eta} = -\frac{C_1}{2\eta}$$

Integrate the optimal control to find the inventory of the trader at time t:

$$x(t) = \int u(t)dt = -\frac{tC_1}{2\eta} + C_2$$

Use the two initial conditions $x(0) = x_0$ and $x(t_1) = x_1$ to solve out the two constants:

$$x(t) = -\frac{t}{t_1}(x_0 - x_1) + x_0 \tag{15}$$
$$t \in (0, t_1)$$

Note that the optimal inventory of the trader is simply a straight line connecting x_0 and x_1.

Optimal path during the fix window

Also for trading in the fix window will it turn out that trader inventory is a linear function of time. The optimal control problem is now

$$\max \left\{ \int_{t_1}^{T} \eta[u(t)]^2 - x_0\eta\frac{1}{T - t_1} \right\}$$

subject to

$$x(t_1) = x_1$$
$$x(T) = 0$$

By repeating exactly the same steps as we did for trading in the pre-fix period, we find the optimal control and integrate to get the optimal inventory:

$$x(t) = -t\frac{C_1}{2\eta} + t\frac{x_0}{2(T - t_1)} + C_2$$

Set $x(T) = 0$ to solve for C_2:

$$C_2 = T\frac{C_1}{2\eta} - T\frac{x_0}{2(T - t_1)}$$

$$\implies x(t) = (T-t)\left[\frac{C_1}{2\eta} - \frac{x_0}{2(T-t_1)}\right]$$

and use $x(t_1) = x_1$ to solve for C_1:

$$C_1 = \frac{2x_1\eta}{T-t_1} + \frac{\eta x_0}{T-t_1}$$

Inserting back into $x(t)$ shows that the optimal inventory is a straight line connecting x_1 and 0:

$$x(t) = (T-t)\left(\frac{x_1}{T-t_1}\right) \tag{16}$$

$$t \in (t_1, T)$$

Optimal allocation

We have established that the optimal inventory is a piece-wise continuous function of time, with a break-point at (t_1, x_1) (see figure 5):

$$x(t) = \begin{cases} x_0 - \frac{t}{t_1}(x_0 - x_1) & \text{if } t \in (0, t_1) \\ (T-t)\left(\frac{x_1}{T-t_1}\right) & \text{if } t \in (t_1, T) \end{cases} \tag{17}$$

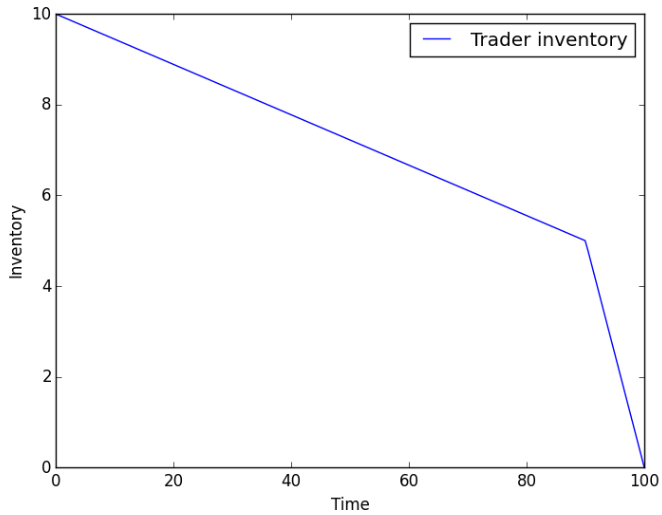


Figure 5: Trader inventory as function of time

It remains for the trader to decide on allocation to the fix window. This choice is determined by setting the inventory level at t_1, namely x_1.

Differentiate equation (17) to get the liquidation intensity $\dot{x}$:

$$\dot{x} = \begin{cases} -\frac{1}{t_1}(x_0 - x_1) & \text{if } t \in (0, t_1) \\ -\frac{x_1}{T-t_1} & \text{if } t \in (t_1, T) \end{cases}$$

Insert $\dot{x}$ back into the maximization problem of the trader (equation 14). Evaluate the integrals - this is easy since $\dot{x}$ is just two constants. The result is a nice expression for $\mathbb{E}\pi$ as a function of x_1 and parameters:

$$\mathbb{E}\pi = \eta \left[-(x_0 - x_1)^2 \frac{1}{t_1} - x_1^2 \frac{1}{T - t_1} + x_0 x_1 \frac{1}{T - t_1} \right]$$

The above expression for expected profit is concave in x_1, so the maximal value can be found by differentiating and setting equal to zero. Doing this and solving for the optimal x_1 yields

$$x_1^* = x_0 \frac{2T - t_1}{2T}$$

Insert back into (17) to characterize the optimal inventory as a function of the model parameters:

$$x(t) = \begin{cases} x_0(1 - \frac{t}{2T}) & \text{if } t \in (0, t_1) \\ \\ x_0 \frac{T-t}{T-t_1} \frac{2T-t_1}{2T} & \text{if } t \in (t_1, T) \end{cases} \tag{18}$$

The time-derivative of (18) is the optimal liquidation intensity:

$$\dot{x} = \begin{cases} -\frac{x_0}{2T} & \text{if } t \in (0, t_1) \\ \\ -\frac{x_0}{T-t_1} \left(1 - \frac{t_1}{2T}\right) & \text{if } t \in (t_1, T) \end{cases} \tag{19}$$

The price distortion from fix orders can be measured as the average difference between the fix price and the fair value in the fixing period:

$$\begin{aligned} &\mid \left(\int_{t_1}^{T} p(s)ds - \int_{t_1}^{T} w(s)ds \right) / (T - t_1) \mid \\ &= \mid \frac{\eta}{T - t_1} \int_{t_1}^{T} \dot{x} ds \mid \\ &= \mid \eta x_0 \frac{1}{T - t_1} \left(1 - \frac{t_1}{2T} \right) \mid \end{aligned} \tag{20}$$

Figure 6 plots equation (20), showing that price distortion is convex and decreasing in the length of the fix period. Lengthening the time window of the fixing period from 1 minute to say 15 minutes would therefore be highly effective in reducing price distortion from traders with fix orders.

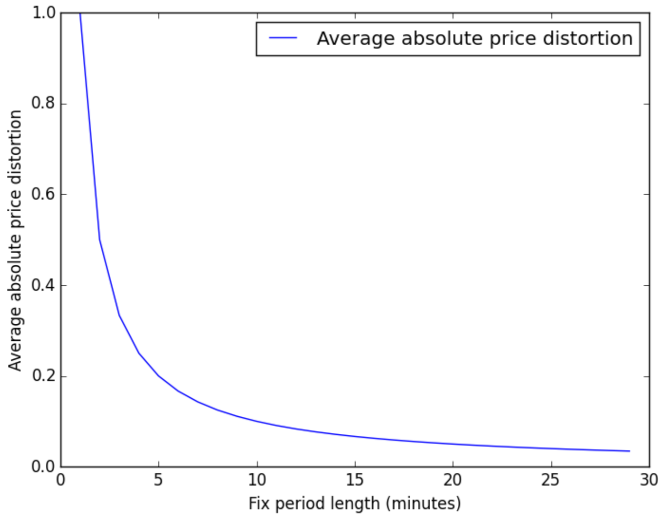


Figure 6: Increasing the length of the fix reduce price distortion significantly ($T = 500$, $\eta = 0.1$, $x_0 = 10$, $t_1 \in [470, 499]$

3 Discussion and policy implications

Is banging the close price manipulation?

To tackle whether or not banging the close is price manipulation, we first consider to related questions: Is banging the close illegal? And *should* it be illegal?

The first of these is clearly outside the scope of the present paper, but a naive approach would be to say that an excess of $11 billion in fines across several jurisdictions indicate that the answer in many cases is in the negative. The second question is more interesting from an economic perspective. One point of view were eloquently stated by Eiteman et al. (1966):

> There is nothing essentially evil about forcing prices up or down. The harm, if any, results from hiding the manipulative activities with intent to deceive investors. The real question is: Would an investor have purchased or sold the manipulated issue if he had known that the price quoted on the Exchange was an artificial one not derived from the free interplay of the forces of supply and demand?

According to this view, banging the close is not a problem in itself. In the model of section 2.3, rational asset managers willingly place fix orders, even though they are fully aware the trader will bang the close. The reason is that asset managers dislike uncertainty in the difference between the benchmark and their transaction price, for instance because they want a low tracking error relative to an index. This is of course not at all saying that real-world investors knew what was going on in the actual FX fixing scandal[7], but at least going forward it would be hard for an asset manager active in FX markets to claim ignorance of the issue.

A more recent contribution related in spirit to that of Eiteman and colleagues were made by Kyle and Viswanathan (2008). They pose two criteria an activity should fulfill in order to be deemed illegal price manipulation: *i)* that it makes prices less accurate as signals for resource allocation, and *ii)* that it makes markets less liquid for risk transfers. While the first criterion may be satisfied for the trading strategy considered here, it is not clear whether the second is. Indeed, one could argue that if many traders are banging the close, it should attract capital willing to absorb the large volumes being traded for a smaller compensation and thereby dampening the transitory price impact. In a perfect market, unsystematic risk factors should not command a risk premium (Cochrane, 2009).

Be that as it may, it seems clear that since the whole idea of banging the close is to cause a large temporary price impact, the strategy creates an artificially inflated (or depressed) price for a short while. This corresponds to the most common definition of market manipulation, here taken from the US Federal Bureau of Investigation:

> Market manipulation is artificially raising or lowering the price of stock on any national securities or commodities exchange or in the over-the-counter (OTC) marketplace.

To sum up, if we take the view that the sharp price spikes induced by traders banging the fix is an artificial price movement, the trading strategy is clearly a form of market manipulation. It also seems clear from the many regulatory cases in different markets and different jurisdictions that the practice is often illegal. It is less clear that the strategy itself hurts market efficiency, and following the views of both Eiteman et al. (1966) and Kyle and Viswanathan (2008) one could argue that regulators should rather be concentrating their efforts elsewhere.

[7]But in all fairness, perhaps investors should have realized something was amiss. In a widely circulated 2010 analysis from Morgan Stanley (*A guide to FX transaction costs, part III*), Paul Aston points out that an international equity portfolio tracking the MSCI and systematically executing at the 4 pm London fix each day, could generate an annualized regret of as much as 0.5 percentage points relative to best execution practice.

More clear-cut is the case of *collusive* banging on the fix. The traders involved in the FX fixing scandal shared privileged client information with each other, with the aim of profiting at the expense of their clients. While front-running client orders is not necessarily prohibited in FX markets, collusive front running usually is.

Policy implications

I would like to point to three policy implications of this study. First, the model shows that banging the close can be an optimal order execution strategy for fix-type orders, and that rational asset managers may choose to place such orders even though they understand the behavior of traders. If regulators find the price spikes and distorted benchmark prices problematic, it seems natural to change the benchmark procedure itself rather than regulate trading strategies.

The second policy implication concerns how to best change the benchmark procedure. Widening the time window of the fix would likely be a highly effective measure in reducing price distortion from fix orders, as shown in section 2.5.

A third implication of the model is that information sharing amongst traders may facilitate collusion, without any need of further coordination or profit-sharing mechanism. By sharing information and thereafter simply trading according to their Nash-equilibrium strategies, the traders of this model can establish a highly stable collusive arrangement. The real-world traders who established chatrooms with ominous sounding names might only be the tip of an iceberg. Information sharing by spreading "rumors" about order flows are common amongst traders in many markets. This may very well be an issue worthy of regulatory scrutiny.

Further research topics and summary

This study opens up several avenues for further research. First, it introduces what is essentially an agency problem into an optimal execution-problem. As the variations of both problem classes are many, there seems to be a potential for much more research in this area. It would for example turn out that the classic arrival price-order in Almgren and Chriss (2001) is also subject to a manipulative execution strategy, if the trader already has an existing order and hence can affect the arrival price of the new order.

A second research topic indicated by the present study is in making a richer game theoretic model of trader behavior when there are benchmark prices that the traders can collectively affect. One modeling approach that seems to have great promise in this setting is global games (Morris and Shin, 2001), as well as considering repeated games between the traders.

Summing up, this paper shows how the introduction of an agency problem into an optimal order execution-model yields new insight into one of the largest market manipulation scandals in modern times. The modeling framework is flexible enough to account for rational investors, games of collusion and the effect of structural changes to the price benchmark. Based on the model, we can derive new policy recommendations to enhance the robustness of financial benchmarks and the efficiency of markets.

References

Alfonsi, Aurélien, Antje Fruth, and Alexander Schied, "Optimal execution strategies in limit order books with general shape functions," *Quantitative Finance*, 2010, *10* (2), 143–157.

Allen, Franklin and Douglas Gale, "Stock-price manipulation," *Review of Financial Studies*, 1992, *5* (3), 503–529.

_ **and Gary Gorton**, "Stock Price Manipulation, Market Microstructure and Asymmetric Information," NBER Working Papers 3862, National Bureau of Economic Research, Inc October 1991.

_ **, Lubomir Litov, and Jianping Mei**, "Large investors, price manipulation, and limits to arbitrage: an anatomy of market corners," *Review of Finance*, 2006, *10* (4), 645–693.

Almgren, Robert F., "Optimal execution with nonlinear impact functions and trading-enhanced risk," *Applied mathematical finance*, 2003, *10* (1), 1–18.

_ , "Optimal trading with stochastic liquidity and volatility," *SIAM Journal on Financial Mathematics*, 2012, *3* (1), 163–181.

_ **and Neil Chriss**, "Optimal execution of portfolio transactions," *Journal of Risk*, 2001, *3*, 5–40.

Bayraktar, Erhan and Michael Ludkovski, "Liquidation in limit order books with controlled intensity," *Mathematical Finance*, 2012.

Bertsimas, Dimitris and Andrew W Lo, "Optimal control of execution costs," *Journal of Financial Markets*, 1998, *1* (1), 1–50.

Bommel, Jos Van, "Rumors," *Journal of Finance*, 2003, *58* (4), 1499–1520.

Cochrane, John H., *Asset Pricing:(Revised Edition)*, Princeton University Press, 2009.

Comerton-Forde, Carole, Alex Frino, and Vito Mollica, "The impact of limit order anonymity on liquidity: Evidence from Paris, Tokyo and Korea," *Journal of Economics and Business*, 2005, *57* (6), 528–540.

_ **and Tālis J Putniņš**, "Measuring closing price manipulation," *Journal of Financial Intermediation*, 2011, *20* (2), 135–158.

_ **and** _ , "Pricing accuracy, liquidity and trader behavior with closing price manipulation," *Experimental Economics*, 2011, *14* (1), 110–131.

Cont, Rama and Arseniy Kukanov, "Optimal order placement in limit order markets," *Available at SSRN 2155218*, 2013.

Copeland, Thomas E. and Dan Galai, "Information effects on the bid-ask spread," *Journal of Finance*, 1983, *38* (5), 1457–1469.

Eiteman, Wilford J., Charles A. Dice, and David K. Eiteman, *The Stock Market*, McGraw-Hill, 1966.

Evans, Martin D.D., "Forex trading and the WMR fix," *Available at SSRN 2487991*, 2014.

Financial Stability Board, "Foreign Exchange Benchmarks," Technical Report, Foreign Exchange Bencmark Group, Financial Stability Board 2014.

Forsyth, Peter A., "A Hamilton–Jacobi–Bellman approach to optimal trade execution," *Applied Numerical Mathematics*, 2011, *61* (2), 241–265.

_ **, J. Shannon Kennedy, S.T. Tse, and Heath Windcliff**, "Optimal trade execution: a mean quadratic variation approach," *Journal of Economic Dynamics and Control*, 2012, *36* (12), 1971–1991.

Frei, Christoph and Nicholas Westray, "Optimal execution of a VWAP order: a stochastic control approach," *Mathematical Finance*, 2013.

Gatheral, Jim, "No-dynamic-arbitrage and market impact," *Quantitative Finance*, 2010, *10* (7), 749–759.

_ **and Alexander Schied**, "Optimal trade execution under geometric Brownian motion in the Almgren and Chriss framework," *International Journal of Theoretical and Applied Finance*, 2011, *14* (03), 353–368.

Glosten, Lawrence R. and Paul R. Milgrom, "Bid, ask and transaction prices in a specialist market with heterogeneously informed traders," *Journal of Financial Economics*, 1985, *14* (1), 71–100.

Hart, Oliver D., "On the profitability of speculation," *Quarterly Journal of Economics*, 1977, pp. 579–597.

Hillion, Pierre and Matti Suominen, "The manipulation of closing prices," *Journal of Financial Markets*, 2004, *7* (4), 351–375.

Ho, Thomas and Hans R. Stoll, "Optimal dealer pricing under transactions and return uncertainty," *Journal of Financial Economics*, 1981, *9* (1), 47–73.

Huberman, Gur and Werner Stanzl, "Optimal liquidity trading," *Review of Finance*, 2005, *9* (2), 165–200.

Jarrow, Robert A., "Market Manipulation, Bubbles, Corners, and Short Squeezes," *Journal of Financial and Quantitative Analysis*, 1992, *27* (3), pp. 311–336.

_ , "Derivative security markets, market manipulation, and option pricing theory," *Journal of Financial and Quantitative Analysis*, 1994, *29* (02), 241–261.

Kissell, Robert, Morton Glantz, and Roberto Malamut, *Optimal trading strategies: quantitative approaches for managing market impact and trading risk*, PublicAffairs, 2003.

Kyle, Albert S., "Continuous auctions and insider trading," *Econometrica*, 1985, pp. 1315–1335.

_ **and S. Viswanathan**, "How to define illegal price manipulation," *American Economic Review*, 2008, pp. 274–279.

Løkka, Arne, "Optimal liquidation in a limit order book for a risk-averse investor," *Mathematical Finance*, 2013.

Melvin, Michael and John Prins, "Equity hedging and exchange rates at the London 4p.m. fix," *Journal of Financial Markets*, 2015, *22* (C), 50–72.

Morris, Stephen and Hyun Song Shin, "Global games: theory and applications," 2001.

Nash, John F., "Equilibrium points in n-person games," *Proceedings of the National Academy of Sciences of the USA*, 1950, *36* (1), 48–49.

Obizhaeva, Anna A. and Jiang Wang, "Optimal trading strategy and supply/demand dynamics," *Journal of Financial Markets*, 2013, *16* (1), 1–32.

Rime, Dagfinn and Andreas Schrimpf, "The anatomy of the global FX market through the lens of the 2013 Triennial Survey," *BIS Quarterly Review, December*, 2013.

Schied, Alexander and Torsten Schöneborn, "Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets," *Finance and Stochastics*, 2009, *13* (2), 181–204.

Seierstad, Atle and Knut Sydsaeter, *Optimal control theory with economic applications*, Elsevier North-Holland, Inc., 1986.

Vayanos, Dimitri, "Strategic trading in a dynamic noisy market," *Journal of Finance*, 2001, pp. 131–171.

Vega, Josef De La, "Confusion de Confusiones: Portions Descriptive of the Amsterdam Stock Exchange.," *(Translation by H. Kellenbenz, Harvard University, 1957)*, 1688.

Vila, Jean-Luc, "Simple games of market manipulation," *Economics Letters*, 1989, *29* (1), 21–26.

Vitale, Paolo, "Speculative noise trading and manipulation in the foreign exchange market," *Journal of International Money and Finance*, 2000, *19* (5), 689–712.