Aspects of Algorithmic and High-Frequency Trading

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Outline

- Exchanges
- Evolution of Markets: the Rise of Algorithmic Trading
- Limit Order Book
- Large Orders and Market Impact
  - Optimal Liquidation/Acquisition
- Market Making
- Market Quality and Regulation

Exchanges

An exchange is a ‘place’ where ‘people’ meet to buy/sell securities: shares, commodities, derivatives, etc.

- Order-Driven Market:
  - All buyers and sellers display the prices and quantities at which they wish to buy or sell a particular security.
- Quote-Driven Market:
  - Designated market makers and specialists display bid and ask prices for a specific security.

Algo Trading

Algorithmic Trading

The use of computer algorithms that make trading decisions, submit orders, and manage those orders after submission

Evolution of markets: why AT (and HFT) now?

A bit of financial regulation.... and other things

- Introduction of competition
  - Market fragmentation
  - Exchanges compete for business
- Direct market access
- US
  - Reg. ATS (Alternative Trading System) – non-exchange venues, for example Dark Pools, operate alongside exchanges
  - Decimalisation 2001, tick sizes decreased
  - Reg. NMS (National Market System) 2005 – Trade Through Rule to ensure that orders are done at the National Best Bid and Offer (NBBO)
- Europe
  - MiFID (Markets in Financial Instruments Directive): harmonised regulation across Europe
  - Does not contemplate Trade Through

Algo and HF Trading

The Rise of Algo Trading

- Large institutional clients need to buy or sell large amounts of shares
- To avoid causing adverse price moves, large orders are:
  - broken into smaller 'child-orders'
  - executed over a period of time (urgency specified by client)
  - performance is measured against a benchmark

Algorithmic Trading

The use of computer algorithms that make trading decisions, submit orders, and manage those orders after submission

- Depending on the objective and parameters, different algorithms are used:
  - VWAP
  - TWAP
  - POV

Algos

- VWAP
  - The objective is to obtain the volume-weighted average price (might be a self-fulfilling prophecy)
  - Must predict the volume
- TWAP
  - The objective is to obtain the time-weighted average price
- POV
  - Use a constant participation rate, say β
  - For example, if the agent still needs to purchase Q shares, the algo computes the volume traded over a time window, say V, and then executes $\min(Q, \beta V)$

High-Frequency Trading

High-Frequency Trading

Refers to the **subset of proprietary algorithmic trading** strategies, which are automated and executed at very high frequencies or very high speed

High-Frequency Trading – Market Making

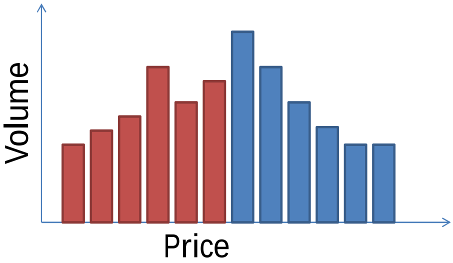
Refers to the **subset of High-Frequency Trading** strategies, which are characterised by their reliance on speed differences relative to other traders to make profits based on short-term predictions and also (objective/consequence) to hold essentially no asset inventories for more than a very short period of time.

Limit Order Book

Limit Order Book

- During the trading day, all **limit orders** are accumulated in the limit order book (LOB) until they find a counterparty for execution or are cancelled by the agent who posted them.
- The counterparty is a **market order**, which is an order to buy or sell an amount of shares, regardless of the price, which is immediately executed against limit orders resting in the LOB at the best execution prices.

LOB

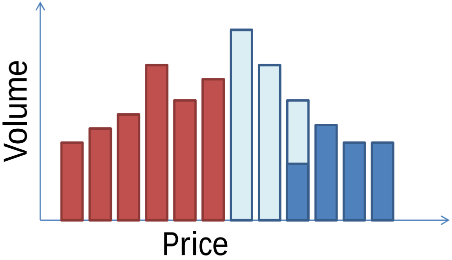


Price/time prioritised collection of buy and sell quotes

Buy Orders		Sell Orders	
Price	Volume	Price	Volume
60	100	60.1	130
59.90	80	60.2	110
59.80	112	60.3	80

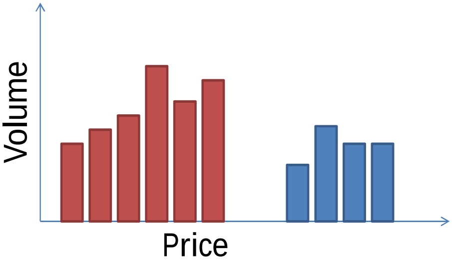
LOB

- **Buy market order arrives**... (sent by slow or fast traders)



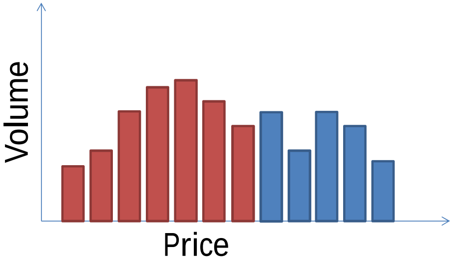
LOB

- Buy market order arrives... **eats part of the book**...



LOB

- **Book recovers** (resiliency)... new limit orders arrive and others are cancelled



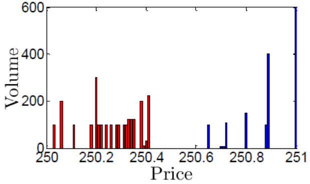
Large Orders and Market Impact

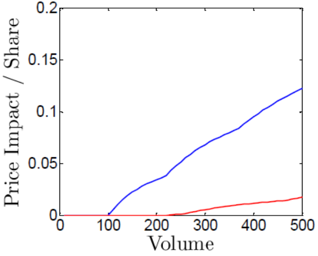
Now imagine you are a **large fund**...

You are a **large** equity fund and must rebalance your portfolio. What are your concerns?

- being 'frontrun' or detected by other clever algorithms
- your order is too large and will move the market against you
- etc

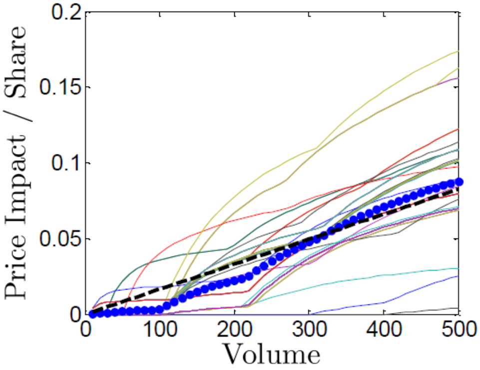
NETFLIX: One day in June 2011, Nasdaq





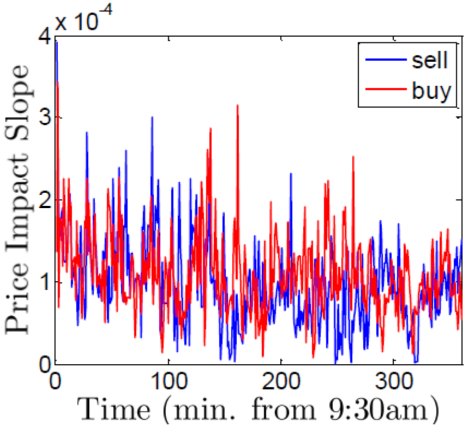
NETFLIX

- The impact every 1sec
- The average impact over 1 minute (using 1sec impact)
- Linear impact approximation (Price Impact = slope)



NETFLIX

- Evolution of impact through the day
  - using 1 minute averages from 1sec impact functions



Optimal Liquidation/Acquisition

$$
\begin{aligned}
dQ_t^{\nu} &= \pm\nu_t\, dt, & Q_0^{\nu} &= q, & &\text{agent's inventory},\\
dS_t^{\nu} &= \pm g(\nu_t)\, dt + \sigma\, dW_t, & S_0^{\nu} &= S, & &\text{fundamental asset price},\\
\hat{S}_t^{\nu} &= S_t^{\nu} \pm h(\nu_t), & \hat{S}_0^{\nu} &= \hat{S}, & &\text{execution price},\\
dX_t^{v} &= \nu_t\, \hat{S}_t^{\nu}\, dt, & X_0^{v} &= x, & &\text{agent's wealth},
\end{aligned}
$$

where $\pm$ indicates whether the agent is liquidating (-) or acquiring (+) shares.

- $\nu = (\nu_t)_{\{0\le t\le T\}}$ liq/acq rate and is what the agent can control,
- $g : \mathbb{R}_+ \to \mathbb{R}_+$ agent's permanent impact on the fundamental price,
- $h : \mathbb{R}_+ \to \mathbb{R}_+$ agent's temporary impact on the price.

Optimal liquidation

$$H(t,x,S,q) = \sup_{\nu \in \mathcal{A}_{t,T}} \mathbb{E}\left[X_T^\nu + Q_T^\nu (S_T^\nu - \alpha Q_T^\nu) - \phi \int_t^T (Q_u^\nu)^2 \, du \right]$$

The DPE then implies that the value function should satisfy the equation

$$\begin{aligned} 0 = &\left(\partial_t + \tfrac{1}{2}\sigma^2 \, \partial_{SS}\right) H(t,x,S,q) - \phi \, q^2 \\ &+ \sup_\nu \left\{ \left(\nu \, (S - h(\nu)) \, \partial_x - g(\nu) \, \partial_S - \nu \, \partial_q \right) H(t,x,S,q) \right\} \end{aligned}$$

subject to the terminal condition

$$H(T,x,S,q) = x + f \, q - \alpha q^2,$$

and with

$$\mathcal{L}^\nu = \nu \, (S - h(\nu)) \, \partial_x - g(\nu) \, \partial_S + \tfrac{1}{2}\sigma^2 \, \partial_{SS} - \nu \, \partial_q \, .$$

Liquidation rate

Assuming that permanent and temporary price impact functions are linear in the speed of trading:

- $h(\nu) = a\,\nu$ for constant $a > 0$,
- $g(\nu) = b\,\nu$ for constant $b > 0$,

and using the Ansatz $H(t, x, S, q) = x + f\,q + h(t, S, q)$ the optimal liquidation rate is

$$\nu_t^* = -\sqrt{a\,\phi}\,\frac{1 + \xi\,e^{2\gamma\,(T-t)}}{1 - \xi\,e^{2\gamma(T-t)}}\,\frac{1}{a}\,q_t\ ,$$

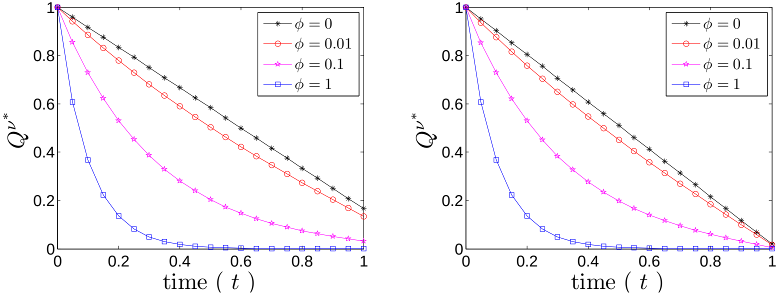
where

$$\gamma = \sqrt{\frac{\phi}{a}} \quad \text{and} \quad \xi = \frac{\alpha + \frac{1}{2}b + \sqrt{a\,\phi}}{\alpha + \frac{1}{2}b - \sqrt{a\,\phi}}\ .$$

Moreover, the inventory along the optimal strategy is given by

$$Q_t^{\nu^*} = Q_0\,\frac{\xi\,e^{\gamma(T-t)} - e^{-\gamma(T-t)}}{\xi\,e^{\gamma\,T} - e^{-\gamma\,T}}\ .$$

Inventory level with $a = 10^{-2}$, $b = 10^{-4}$



(c) $\alpha = 0.05$ (d) $\alpha = 0.5$

- If $\phi = 0$ the strategies are straight lines
- If ϕ is ‘large’ curves become more convex and the optimal strategy aims to sell more assets sooner
- If $\alpha \to \infty$ the strategy is equivalent to one which would give a cost equal to the time weighted average price (TWAP)

Market Maker (Slow or Fast)

Imagine that you are a market maker... (part I)

- How do you make money in this business?
- What information do you use?
- How can you devise a strategy to make money?
- Are there any events that make you withdraw from the market?

Imagine you are a market maker... (part II)

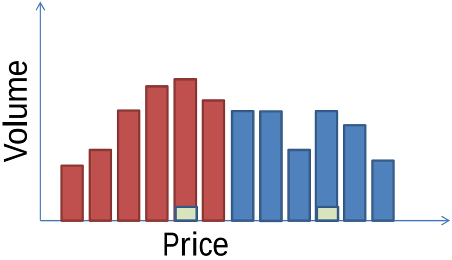
- You can observe the entire LOB
- You do not know who you trade with
- Make money from roundtrip trades by posting limit buy and limit sell orders (maybe you also send market orders)
- Must be prepared to hold inventories

What is a market making strategy?
It may mean many things, for instance:

- Where to place your limit orders
- Are you posting in many places in the book?
- How do you react to **order-flow** information?
- When is it good to cancel an existing limit order?
- How do you manage inventory risk?
  - How much is too much or too little?
  - When do you aggressively reduce your inventory risk?

LOB

- HFT's or MM's **add limit orders** ...



- HFTs also send market orders

Model

- Assume that $S(t)$ satisfies
$$dS(t) = \sigma dW(t)$$
- The MM places optimal buy limit order $S_t - \delta_t^-$ and a sell limit order $S_t + \delta_t^+$ (with $\delta_t^\pm > 0$) in the LOB.
- Arrival of sell and buy market orders are modelled by Poisson processes M_t^- and M_t^+, respectively, with intensities λ^- and λ^+.
- The MM's filled orders are denoted by the counting processes $(N_t^\pm)_{0\le t\le T}$.
- The HF trader's wealth X_t satisfies
$$dX_t = (S_t + \delta_t^+)\, dN_t^+ - (S_t - \delta_t^-)\, dN_t^-\,,$$
and the investor seeks the strategy $(\delta_s^\pm)_{t\le s\le T}$.

MM's Control Problem

The MM trader solves the control problem
$$H(t,x,q,S) = \sup_{(\delta_s^\pm)_{t\le s\le T}\in\mathcal{A}} \mathbb{E}\left[X_T + q_T(S_T - \alpha\, q_T)\right]\,.$$

HJB

The DPE then implies that the value function should satisfy the equation

$$
\begin{aligned}
0 \quad = \quad & \partial_t H + \tfrac{1}{2}\sigma^2 \partial_{SS} H \\
& + \lambda^+ \sup_{\delta^+} \left\{ e^{-\kappa^+ \delta^+} \left(H(t, x + (S + \delta^+), q - 1, S) - H \right) \right\} \\
& + \lambda^- \sup_{\delta^-} \left\{ e^{-\kappa^- \delta^-} \left(H\left(t, x - (S - \delta^-), q + 1, S\right) - H \right) \right\} ,
\end{aligned} \tag{2}
$$

with terminal condition

$$
H(T, x, q, S) = x(T) + q(T)(S(T) - \alpha q(T)) . \tag{3}
$$

Assume $\alpha = 0$

$$H(t, x, q, S) = x + qS + g(q)\,.$$

Note that the function $g(q)$ does not depend on time. Thus

$$0 = \lambda^+ \sup_{\delta^+} \left\{ e^{-\kappa^+ \delta^+} \, \delta^+ \right\} + \lambda^- \sup_{\delta^-} \left\{ e^{-\kappa^- \delta^-} \, \delta^- \right\} \,,$$

and the optimal postings are:

$$\delta^{*,+} = \frac{1}{\kappa^+}$$

and

$$\delta^{*,-} = \frac{1}{\kappa^-}\,.$$

Note that with $\alpha = 0$ the agent maximises expected half-spreads

Market Quality and Regulation!!

Has the marketplace improved?

- This is a rather tricky question!
- What is a relevant measure of market quality?
  - Spreads
  - Effective spread
  - Market depth
  - Volatility
  - Price discovery

Empirical evidence

- SEC-CFTC 'Concept Release on Equity Market Structure' on (2010) Flash Crash
- Hasbrouck and Saar (2011)
  - find that low-latency activity improves market quality (effective spread, range of the mid-point, displayed depth)
- Cartea, Payne, Penalva, Tapia (2013)
  - Ultra-fast trading
    - decreases depth in the LOB,
    - increases the effective spread,
    - increases quoted spread.

What is wrong with HFT?

Regulators have thought about the following

- Minimum resting times
- Market-making obligation
- Executions to messages ratio
- Tick sizes
- Circuit breakers
- "Can we have your algos, please?"
- Market making obligation
- 'Call auctions' and 'continuous trading'

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