Jump Diffusion, Mean and Variance: How to Dynamically Hedge, Statically Hedge and to Price

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Abstract
We consider mean-variance analysis so as to propose improvements, or at least "things to think about," in the classical problem of jump-diffusion option pricing. We see how to minimize variance by dynamically hedging, and then how to further reduce jump risk by static hedging. Results are given for a up-and-out barrier call option.

Keywords
Jump; Poisson process; Option pricing; Mean-variance pricing; Risk minimization; Static hedging

1 Introduction

In Ahn & Wilmott (2003) we wrote about mean-variance pricing in a stochastic volatility framework to solve the problem of hedging and pricing in incomplete markets. In the interests of completeness, in modelling if not in markets,[1] we are here going to give the details of applying the same technique but now in a jump-diffusion setting. See Merton (1976) for the original work on jump-diffusion models. Other work of interest in this field is due to Platen (2004), Arai (2005), Lin (2005) and Cont, Tankov & Voltchkova (2006) who variously consider incomplete markets, including mean-variance pricing and utility theory. Tankov (2007) also considers various dynamic hedging strategies with some *ad hoc* static hedging.

Our proposals result in non-linear models for the value of options. One should therefore see the work by Avellaneda & Parás (1996) and Hua & Wilmott (1997) on other non-linear models (uncertain volatility and crashes), summarized in Wilmott (2006).

2 The Model for the Underlying Asset

We are going to work with the classical jump-diffusive random walk for an asset, S, given by

$$dS = \mu S\,dt + \sigma S\,dX + (J-1)S\,dq,$$

where μ is the drift rate in the absence of jumps, σ is the volatility in the absence of jumps, dX is a Wiener process, J is the, possibly random, factor representing the size of the jump (so that a stock that has value S before a jump becomes JS after a jump), dq is a Poisson process with intensity λ. We assume that all of the parameters in the above and the distribution of J are known. They need not be constant since we will typically have to solve our final equations numerically, they may therefore be time and/or asset dependent.

3 Processes

The Poisson process q is independent of the Wiener process X and $P(dq=0) = 1-\lambda dt + o(dt)$, $\quad P(dq=1) = \lambda dt + o(dt)$, and $P(dq>1) = o(dt)$. In addition, the jump size J is independent of both q and X.

Given a function f with suitable differentiability and integrability conditions, we have

$$\begin{aligned} &E_t\left[f(S+dS, t+dt) - f(S,t)\right] \\ &= E_t\left[f(S+\mu dt + \sigma S dX, t+dt) - f(S,t)\right](1-\lambda dt) \\ &\quad + E_t\left[f(\mu dt + \sigma S dX + JS, t+dt) - f(S,t)\right]\lambda dt + o(dt) \end{aligned}$$

$$= f_t dt + f_S \mu S\, dt + \frac{1}{2} f_{SS} \sigma^2 S^2 dt + \lambda E\left[f(JS,t) - f(S,t)\right] dt + o(dt)$$

4 Concept

In what follows there will be no talk of risk neutrality, risk-neutral probabilities or risk-neutral distributions. All processes are real and all expectations are real. The symbol $E[\cdot]$ will be used to denote the real expectation over J, while $E_t[\cdot]$, as used above, is over dX, dq and J.

The concept described in this paper is as follows.

- We want to replicate an option as closely as possible using a dynamic strategy in the underlying asset, Δ
- Because of the existence of jumps, this replication will not be perfect. Our market is incomplete
- We will consider two dynamic hedging strategies, both highly classical, but different
- The mean of a discounted and aggregated future cash-flow generated from maintaining a dynamic trading strategy Δ is going to be denoted by $m(S, t)$ and its variance by $v(S, t)$
- We will relate these two functions to the 'value' of an option
- We will see how to improve the 'value' of an option by statically hedging

5 Set Up

The evolution of the discounted and aggregated future cash-flow generated from maintaining a dynamic trading strategy Δ is described by:

$$C(S,t) = e^{-rdt} C(S+dS, t+dt) - \Delta dS + r\Delta S dt.$$

The quantity above depends upon the realization of S in the future as well as Δ, and hence is generally not determined at time t, i.e. it is random. The mean and the variance are:

$$m(S,t) = E_t\left[C(S,t)\right],$$
$$v(S,t) = E_t\left[(C(S,t) - m(S,t))^2\right],$$

where E_t, as mentioned, is the expectation at time t.

We will choose Δ so as to replicate an option payoff (or portfolio of options) as closely as possible.

5.1 Mean

From the definition, we have

$$\begin{aligned} m(S,t) &= E_t\left[e^{-rdt} C(S+dS, t+dt) - \Delta dS + r\Delta S dt\right] \\ &= E_t\left[e^{-rdt} m(S+dS, t+dt) - \Delta dS + r\Delta S dt\right]. \end{aligned}$$

Applying the expansion yields

$$\begin{aligned} & m_t + \frac{1}{2}\sigma^2 S^2 m_{SS} + \mu S\, m_S + \lambda E[m(JS,t) - m] \\ & - (\mu - r + \lambda E[J-1]) S\Delta - rm = 0. \end{aligned} \tag{1}$$

In this and unless otherwise specified m is simply $m(S, t)$.

5.2 Variance

Note that

$$\begin{aligned} & C(S,t) - m(S,t) \\ &= e^{-rdt}\left[C(S+dS,t+dt) - m(S+dS,t+dt)\right] \\ &\quad - \Delta dS + r\Delta\, S dt + e^{-rdt} m(S+dS,t+dt) - m(S,t) \end{aligned}$$

The expected value of the square of the first term at t is:

$$e^{-2rdt} E_t\left[V(S+dS, t+dt)\right].$$

The expected value of the cross product of the two terms at t is zero, because the expected value of the first term at $t + dt$ vanishes. The expected value of the square of the second term at t is

$$\begin{aligned} & E_t\left[\left(-\Delta\, dS + r\Delta S\, dt + e^{-rdt} m(S+dS,t+dt) - m(S,t)\right)^2\right] \\ &= \sigma^2 S^2 (m_S - \Delta)^2 dt + \lambda E\left[\left(m(JS,t) - m - (J-1)SdS\right)^2\right] dt \\ &\quad + o(dt). \end{aligned}$$

Thus,

$$\begin{aligned} v(S,t) &= e^{-2rdt} E_t\left[v(S+dS,t+dt)\right] \\ &\quad + \sigma^2 S^2 (m_S - \Delta)^2 dt + \lambda E\left[\left(m(JS,t) - m - (J-1)S\, dS\right)^2\right] dt \\ &\quad + o(dt) \end{aligned}$$

and hence,

$$\begin{aligned} & v_t + \frac{1}{2}\sigma^2 S^2 v_{SS} + \mu S\, v_S + \lambda E[v(JS,t) - v] - 2rv \\ & + \sigma^2 S^2 (m_S - \Delta)^2 + \lambda E\left[(m(JS,t) - m - (J-1)S\Delta)^2\right] = 0. \end{aligned} \tag{2}$$

6 Choosing the Hedge Ratio

The instantaneous variance in this replication portfolio is given by

$$\sigma^2 S^2 (m_S - \Delta)^2 + \lambda E\left[(m(JS,t) - m - \Delta(J-1)S)^2\right], \tag{3}$$

multiplied by dt.

Traditionally in the classical Merton jump model one chooses Δ to remove the diffusive component of the process, leaving only the jump component. The argument is then that the remaining jump risk is not priced in on the grounds of diversification. Of course, this is highly unsatisfactory in theory. In practice, however, it does at least result in some simple closed-form expressions for the value of vanilla options.

In the present framework this would amount to choosing

$$\Delta = m_S. \tag{4}$$

This leaves an instantaneous variance of

$$\lambda E\left[(m(JS,t) - m - (J-1)S\, m_S)^2\right].$$

We call this ‘diffusion-eliminating dynamic hedging.’

However, we can do better than this.

A more satisfactory, albeit more complicated, suggestion (discussed *en passant* in Wilmott, 1998) is to choose Δ to minimize (3). This results in the choice

$$\Delta = m_S + \lambda \frac{E[(J-1)(m(JS,t)-m)-(J-1)^2 S\, m_S]}{S(\sigma^2 + \lambda E[(J-1)^2])}, \tag{5}$$

as can be seen by differentiating (3) with respect to Δ and setting the resulting expression equal to zero.

This leaves an instantaneous variance of

$$\lambda E\left[(m(JS,t) - m - (J-1)S\, m_S)^2\right] - \frac{\lambda^2}{\sigma^2 + \lambda E[(J-1)^2]}(E[(J-1)(m(JS,t)-m)-(J-1)^2 S\, m_S])^2.$$

We call this ‘optimal dynamic hedging.’

We will explore both of these dynamic replication strategies below.

7 Final Conditions, etc.

The equation for the mean is given by (1) subject to the usual final condition depending on the payoff. Δ will be given by either expression (4) or (5) depending on how the dynamic hedge is chosen.

For a call option the final condition is

$$m(S,T) = \max(S-E, 0)$$

where E is the option’s strike and T its expiration.

The equation for the variance is given by (2). Again $v = v(S,t)$, and Δ will be given by either expression (4) or (5) depending on how the dynamic hedge is chosen. This equation is subject to the final condition of zero at expiration.

8 Special Case: Jump to Zero

The following results are all based on the simple case in which the jump factor is zero, that is $J=0$, so that a jump amounts to a total collapse in the value of the underlying asset. This is the simplest case to consider, and has only the one parameter λ, whereas the general model has a λ and a probability distribution for J.[2] In this special case the probability density function for J is a delta function. Let us here emphasize that all of the ideas and qualitative results carry over to the general case of arbitrary jump distributions.[3] The sole difference between the general and the specific is in the complexity of the numerical solution.

This case is particularly easy to analyze since a) there are closed-form formulæ for simple contracts and b) the numerical analysis does not require full solution of a partial *integro*-differential equation. Most comments and results are relevant to the full, general case, of course.

In this special case, the two governing equations are

$$m_t + \frac{1}{2}\sigma^2 S^2 m_{SS} + \mu S m_S - (\mu - r - \lambda)S\Delta - (r+\lambda)m + \lambda m(0,t) = 0, \tag{6}$$

and

$$v_t + \frac{1}{2}\sigma^2 S^2 v_{SS} + \mu S v_S - (2r+\lambda)v + \sigma^2 S^2 (m_S - \Delta)^2 + \lambda(m(0,t) - m + S\Delta)^2 = 0, \tag{7}$$

since it is clear that $v(0,t) = 0$.

The choices for Δ are

$$\Delta = m_S$$

and

$$\Delta = m_S + \lambda \frac{(m - m(0,t) - S m_S)}{S(\sigma^2 + \lambda)}.$$

8.1 Diffusion-eliminating Dynamic Hedging

With the choice $\Delta = m_S$ the equation, (6), for the mean becomes identical to the classical Black–Scholes equation with interest rate, r, replaced by $r+\lambda$ with an added ‘source’ term $\lambda m(0,t)$ on the left-hand side.

8.2 Optimal Dynamic Hedging

With the choice $\Delta = m_S + \lambda \frac{(m-m(0,t)-Sm_S)}{S(\sigma^2+\lambda)}$ the equation, (6), for the mean also becomes identical to the classical Black–Scholes equation with interest rate, r, replaced by

$$r + \lambda + \frac{\lambda}{\sigma^2+\lambda}(\mu - r - \lambda) = r + \frac{\lambda}{\sigma^2+\lambda}(\mu - r + \sigma^2),$$

and this time a source term of $\lambda(\mu - r + \sigma^2)/(\sigma^2+\lambda)\, m(0,t)$.

So, in either case, clearly Black–Scholes-type formulæ will be relevant (and important in checking numerical solutions).

9 Results for a Single Vanilla Option

In this section we look at the valuation of a single call option. (Later we will consider a barrier option.) For a call option we have $m(0,t) = 0$. With the above assumption for the jump the problem for the call options becomes

$$m_t + \frac{1}{2}\sigma^2 S^2 m_{SS} + \mu S m_S - (\mu - r - \lambda)S\Delta - (r+\lambda)m = 0, \tag{8}$$

and

$$v_t + \frac{1}{2}\sigma^2 S^2 v_{SS} + \mu S v_S - (2r+\lambda)v + \sigma^2 S^2(m_S - \Delta)^2 + \lambda(-m + S\Delta)^2 = 0. \tag{9}$$

The choices for Δ are

$$\Delta = m_S$$

and

$$\Delta = m_S + \lambda \frac{(m - Sm_S)}{S(\sigma^2 + \lambda)}.$$

9.1 Diffusion-eliminating Delta

As mentioned above, with the choice $\Delta = m_S$ the equation, (8), for the mean becomes identical to the classical Black–Scholes equation with interest rate, r, replaced by $r + \lambda$.

Figure 1 shows the value of a call option against *strike* both with and without the jump. The variables and parameters are $S = 100$, $\sigma = 0.2$, $r = 0.05$, $T = 0.25$, and $\lambda = 0.05$. (The last parameter is, of course, only relevant in the case of jumps.) Figure 2 shows the same results in terms of implied volatility. (The non-jump case is not shown, it would be a constant 0.2.) Inevitably one finds a negative skew when the call prices are interpreted in terms of implied volatility.

Figure 3 shows the standard deviation, the square root of the variance, of the call option value against strike.

9.2 Optimal Delta

Again, as mentioned above, with the choice $\Delta = m_s + \lambda \frac{(m - Sm_s)}{S(\sigma^2+\lambda)}$ the equation, (8), for the mean also becomes identical to the classical Black–Scholes equation with interest rate, r, replaced by

$$r + \lambda + \frac{\lambda}{\sigma^2 + \lambda}(\mu - r - \lambda) = r + \frac{\lambda}{\sigma^2 + \lambda}(\mu - r + \sigma^2).$$

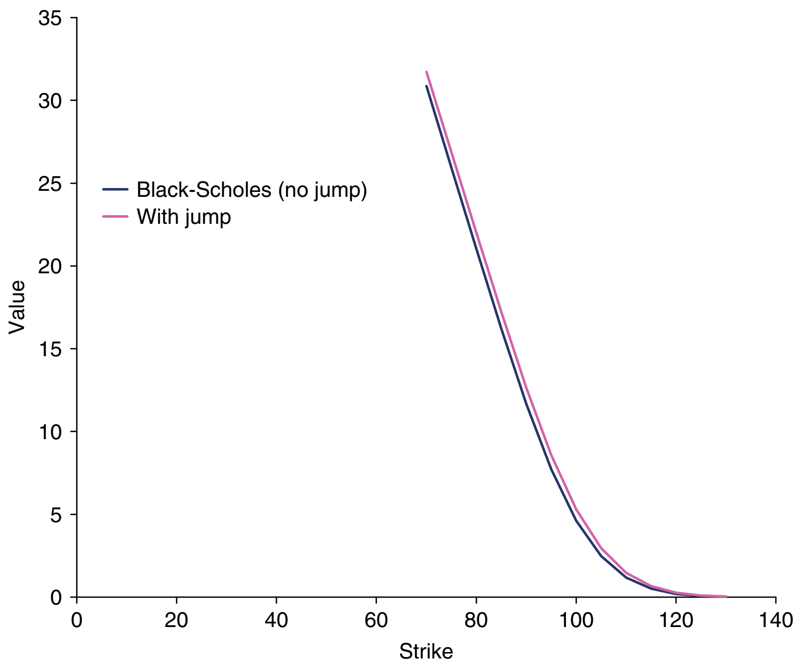

Figure 1: Diffusion-eliminating delta: Call option value versus strike, with and without jumps. $S = 100$, $\sigma = 0.2$, r = 0.05, T = 0.25, and $\lambda = 0.05$.

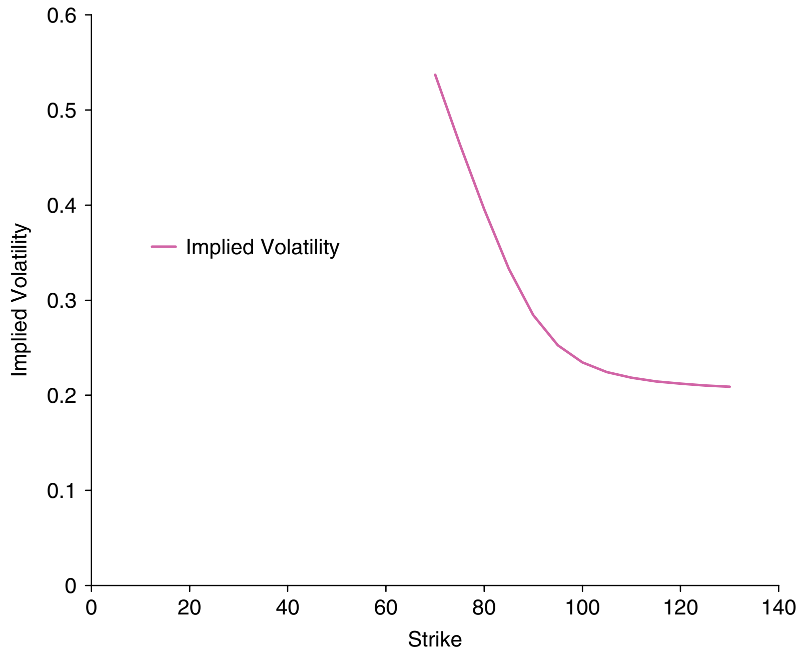

Figure 2: Diffusion-eliminating delta: Implied volatility versus strike, with jumps. $S = 100$, $\sigma = 0.2$, $r = 0.05$, $T = 0.25$, and $\lambda = 0.05$.

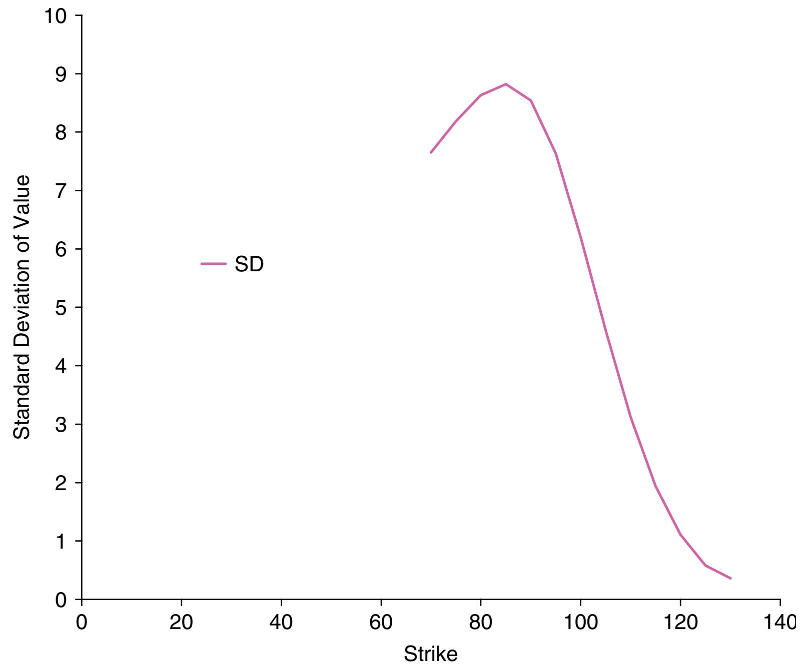

Figure 3: Diffusion-eliminating delta: Standard deviation of call option value versus strike, with jumps. $S = 100$, $\sigma = 0.2$, r = 0.05, $T = 0.25$, and $\lambda = 0.05$.

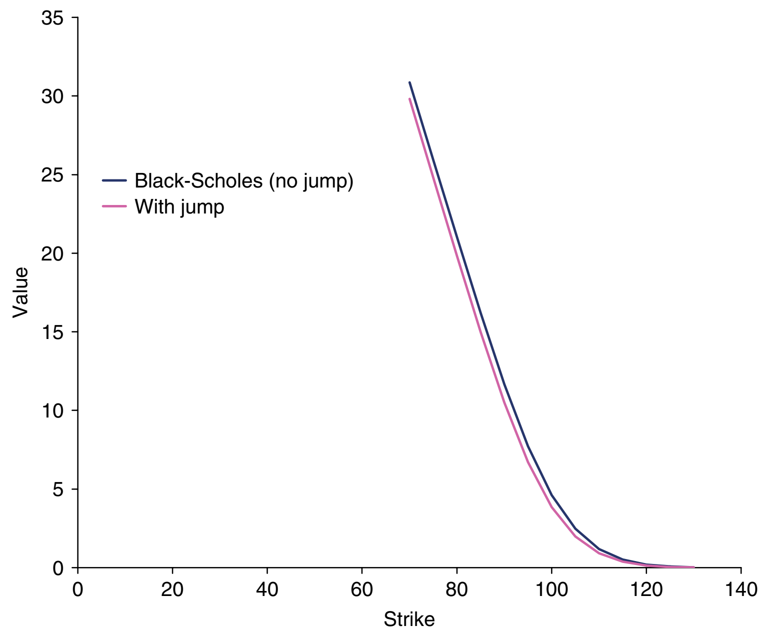

Figure 4: Optimal delta: Call option value versus strike, with and without jumps. $S = 100$, $\sigma = 0.2$, $r = 0.05$, $T = 0.25$, $\lambda = 0.05$ and $\mu = -0.1$.

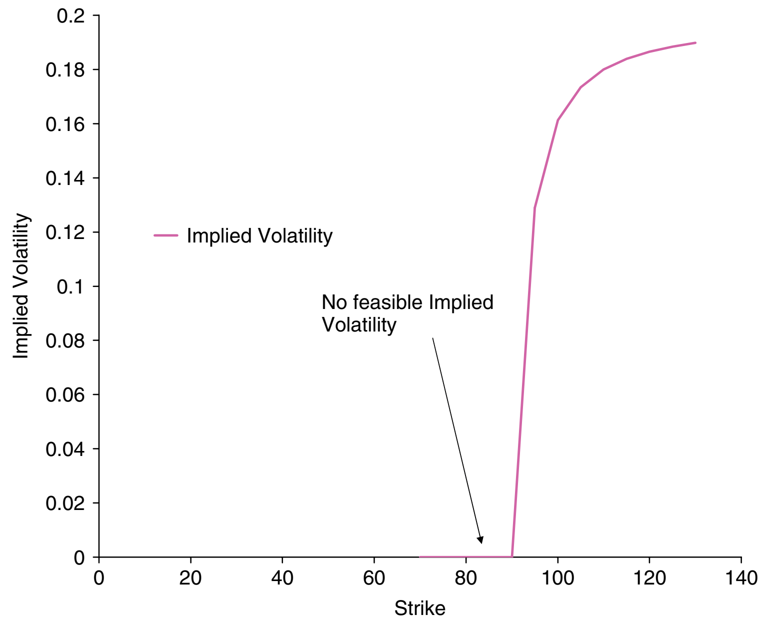

Figure 5: Optimal delta: Implied volatility versus strike, with jumps. $S = 100$, $\sigma = 0.2$, $r = 0.05$, $T = 0.25$, $\lambda = 0.05$ and $\mu = -0.1$.

There are several points of note about this.

1. The value, m, and hence the direction of the skew, depends on μ
2. The skew will be negative if $\mu - r + \sigma^2 > 0$, otherwise positive

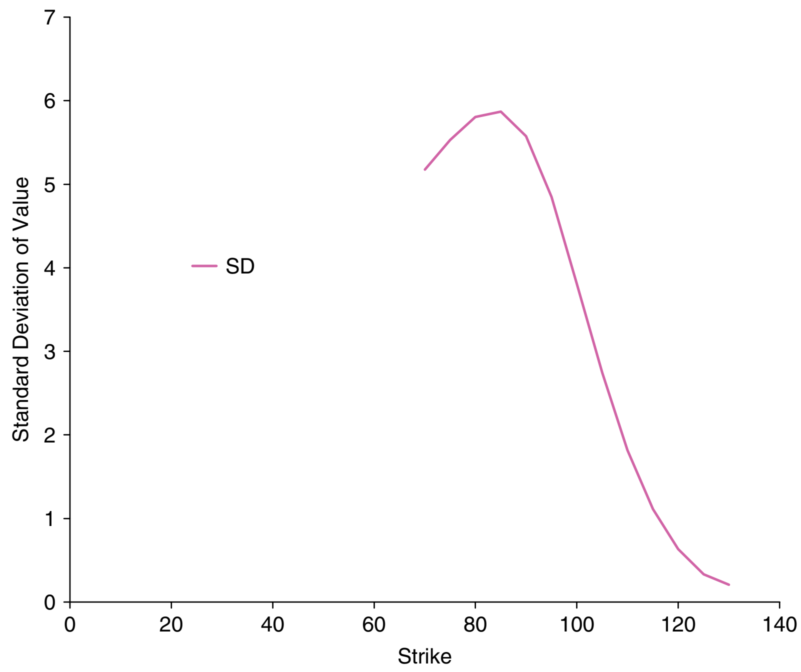

Figure 6: Optimal delta: Standard deviation of call option value versus strike, with jumps. $S = 100$, $\sigma = 0.2$, $r = 0.05$, $T = 0.25$, $\lambda = 0.05$ and $\mu = -0.1$.

3. The skew will be more negative than in the diffusion-eliminating case if $\mu - r - \lambda > 0$

Figure 4 shows the value of a call option against strike both with and without the jump. The variables and parameters are $S = 100, \sigma = 0.2$, $r = 0.05, T = 0.25, \lambda = 0.05$ and $\mu = -0.1$. Figure 5 shows the same results in terms of implied volatility. We have deliberately chosen the parameters to show the atypical positive skew. Indeed, for sufficient low strikes the value goes below the payoff and is therefore inconsistent with any implied volatility.

Figure 6 shows the standard deviation, the square root of the variance, of the call option value against strike. Note that for these parameters the residual risk, as measured by the standard deviation, is half that when diffusion-eliminating dynamic hedging is used.

10 Significance for Local Volatility Models

If, briefly, we interpret the mean, m, as the value of the option then the above results may have some significance for those who either like to interpret prices and/or deltas in terms of implied volatility or who use local volatility models.

Using the diffusion-eliminating dynamic hedge then there are no problems. In our special, jump-to-zero, case the mean is the same as Black–Scholes with an increased interest rate. The effect of the jump to zero is then like having a call with a slightly lower strike. Implied volatilities exist and you get the classical negative skew.

However, with the optimal dynamic hedge the mean is still given by Black– Scholes with an adjusted interest rate, but now the rate could be reduced. And so, depending on the parameters, it would be like having a higher strike for the call. In the European case this can result in values that have no implied volatility. The same is also true of the implied delta that is found by matching, not the price to the market, but the delta to the market delta. In other words the optimal dynamic hedge may give deltas that no Black–Scholes model could give for any volatility.

A local volatility model obviously permits a broader range of prices but similar results could be expected, the best, i.e. optimal dynamic, hedge may not be given even by local volatility models.

11 How to Interpret and Use the Mean and Variance

Take an option position in a world with jumps, and dynamically delta hedge using one of the methods explained above. Because we cannot eliminate all the risk we cannot be certain how accurate our hedging will be. Think of the final value of the portfolio together with accumulated hedging as being the 'outcome.' We can use the mean, m, and variance, v, to help us 'price' the option.

If the distribution of the outcome were normal, which, of course, it won't be, then the mean and the variance are sufficient to describe the probabilities of any outcome. If we wanted to be 95% certain that we would make money then we would have to sell the option for

$$m + 1.644853v^{1/2}$$

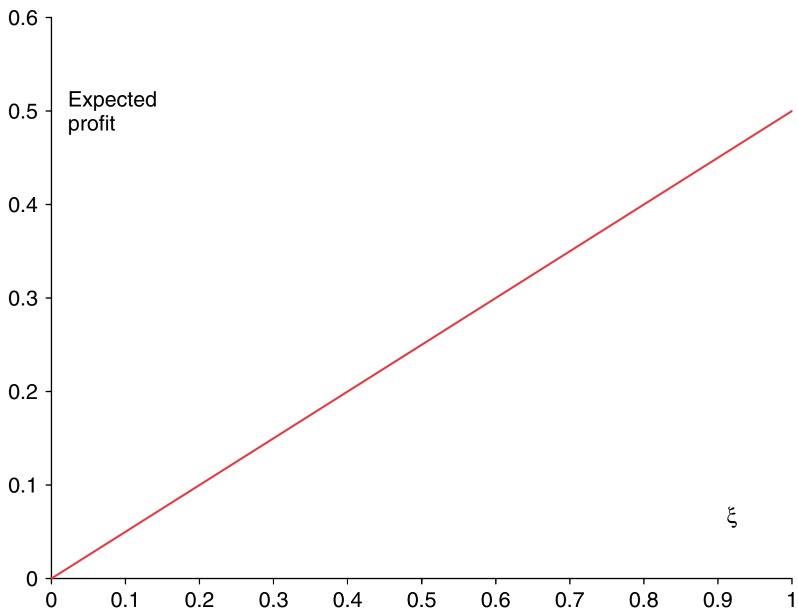

Figure 7: Expected profit from a single trade versus ξ.

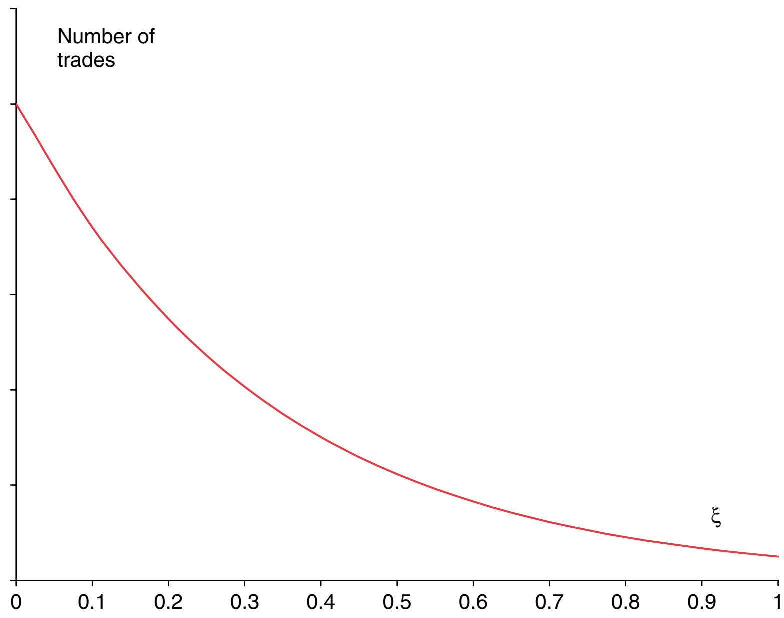

Figure 8: Number of trades versus ξ.

or buy it for

$$m - 1.644853v^{1/2}.$$

The 1.644853 comes from the position of the 95th percentile in a normal distribution.

Of course, the distribution of the outcome will not be normal. The shape will depend very much on the option position we are hedging. However, assuming that our trade is not the only one we have on our books then we can appeal to the Central Limit Theorem to argue that we can still use

$$m \pm \xi v^{1/2},$$

as our price for the option. Here the ξ is a personal choice, representing our degree of risk aversion.

Clearly the larger ξ the greater the potential for profit from a single trade, see Figure 7.

However, the larger ξ the fewer trades, see Figure 8.

The net result is that the total profit potential, being a product of the number of trades and the profit from each trade, is of the form shown in Figure 9. Don't be too greedy or too generous.

12 Nonlinearity and Static Hedging

If we use the above approach to give our option a 'value' then clearly the model is non linear. Nonlinearity is seen in other derivatives models, such as the uncertain volatility model of Avellaneda & Parás (1996), the

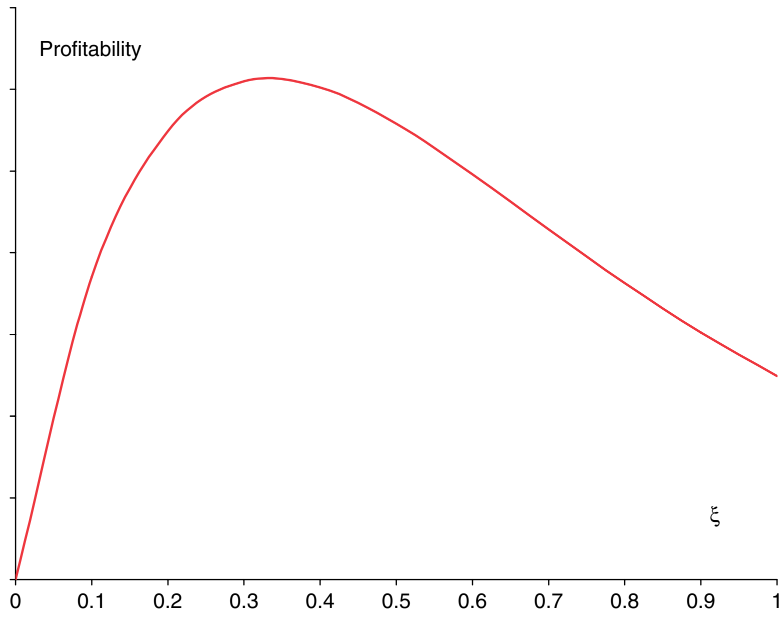

Figure 9: Total profit potential versus ξ.

crash model of Hua & Wilmott (1997) and the stochastic volatility and mean variance model of the present authors (Ahn & Wilmott, 2003). In such non-linear models we typically find that the value of a portfolio made up of several different contracts is different from the sum of the values of the component options valued in isolation. Thus we find economies of scale and the concept of optimal static hedging. To give a simple example of the latter result let us suppose we have an exotic option that we value at $10. This option might have a lot of model risk so we might like to hedge with a vanilla option that has similar risk *but of the opposite sign*. So now suppose we see such an option and it is selling for $5. We buy this option and now value our portfolio of exotic plus vanilla. In a linear world the portfolio would be worth $10 + $5 = $15. So there is no theoretical gain from such static hedging. But in a non-linear world we may find that the new portfolio is worth $17. We have made $2 it appears. Or we could say that the exotic on its own is worth just $10, since it is quite risky. But hedged with a vanilla we reduce risk and so the exotic is worth $17 − $5 (the cost of the static hedge) = $12.

We will see this happen in our barrier option example that follows.

13 Example 1: Valuing and Hedging an Up-and-out Call, No Static Hedge

We are now going to look at the pricing and hedging of a short up-and-out call option. (Because it is a short position, watch out for otherwise unexpected minus signs.) The option has a strike at 100, a barrier at 120 and three months until expiration. The volatility is 20%, drift rate 0%, λ is 5% and the risk-free rate is also 5%. The following are the results of finite-difference schemes, one for each of the m and v equations, in the case of the barrier option valued in isolation.

The problem is as follows. We solve the equations (6) and (7) subject to

(a) $m(S_u, t) = v(S_u, t) = 0$;
(b) $m(S, T) = -\max(S - E, 0)$ where E is the strike;
(c) $v(S, T) = 0$.

In this the barrier is located at S_u.

Notice in the above we are now valuing at expiration the barrier option as $-\max(S - E, 0)$ with the minus sign representing the *sale* of the barrier option.

13.1 Diffusion-eliminating Dynamic Hedge, No Static Hedge

When delta is chosen to eliminate diffusion in the portfolio's value we find that the mean, m, when the underlying has value 100, is -3.34. The variance, v, is 21.85, so that the standard deviation is 4.67. If we were selling this option then we would sell it for

$$-3.34 - \xi \times 4.67.$$

Let us suppose that our personal degree of risk aversion is given by $\xi = 0.5$. Therefore we would sell the option for 5.68 and no less.

13.2 Optimal Dynamic Hedge, No Static Hedge

When delta is chosen optimally we find

$$m = -3.02, \quad \text{and} \quad v = 9.58$$

so we would sell for 4.57. Observe how much the variance has been reduced using the optimal delta. Note also that this is less than the price we would have to sell at in the diffusion-eliminating case. This is not always the situation since the optimally hedged m could be greater than the diffusion-eliminating m.

See Figures 10 and 11 for means and standard deviations against the underlying asset for these two cases.

The above results are shown in the table. DEDH means "Diffusion-eliminating dynamic hedge," "NSH is "No static hedge" and ODH is "Optimal dynamic hedge."

	Mean *(m)*	Var. *(v)*	Value
DEDH, NSH	−3.34	21.85	−5.68
ODH, NSH	−3.02	9.58	−4.57

14 Example continued: Valuing and Hedging an Up-and-out Call, Static Hedge

No let us introduce into our universe the following six call options, all with the same expiration as our barrier option.

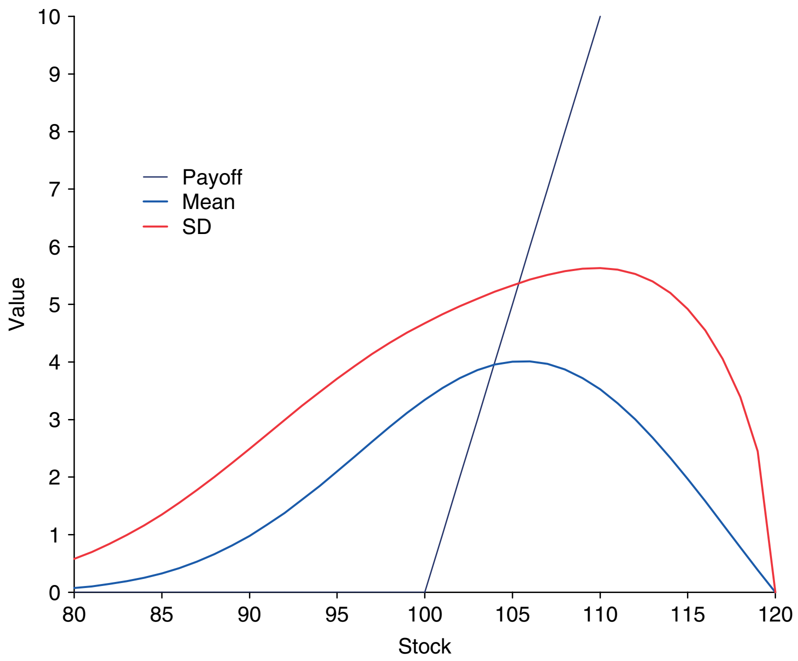

Figure 10: Mean and standard deviation for the diffusion-eliminating hedged up-and-out call barrier option. See text for parameters.

Option	1	2	3	4	5	6
Strike	80	90	100	110	120	130
Bid Price	21.01	11.50	4.22	0.91	0.11	0.01
Ask Price	21.05	11.87	5.01	1.50	0.32	0.02

How can we use these vanillas or the information contained within these prices? We are going to see how to incorporate quantities of the vanillas into a portfolio along with the barrier option to construct a portfolio that has a better 'value' than the barrier on its own, unhedged as above. Here 'value' means the theoretical value of the portfolio under this non-linear model after the cost of the static hedge, the vanillas, has been subtracted off.

14.1 The Algorithm

Suppose we trade $(q_1, \dots, q_6)$ of the above instruments and let E_i be the strikes among the payoffs. Furthermore, let $(m^{(0)}, v^{(0)})$ be the mean variance pair *after* knockout and $(m^{(1)}, v^{(1)})$ be that *before* knockout. Then $(m^{(i)}, v^{(i)})$, $i = 0, 1$, satisfy the equations (6) and (7) subject to:

(a) $m^{(1)}(S_u, t) = m^{(0)}(S_u, t)$ and $v^{(1)}(S_u, t) = v^{(0)}(S_u, t)$;
(b) $m^{(0)}(S, T) = \sum_{i=1}^{6} q_i \max(S - E_i, 0)$;
(c) $m^{(1)}(S, T) = \sum_{i=1}^{6} q_i \max(S - E_i, 0) - \max(S - E, 0)$;
(d) $v^{(1)}(S, T) = v^{(0)}(S, T) = 0$.

Thus $m^{(1)}(S, 0)$ stands for the mean of the cashflows excluding the upfront premium.

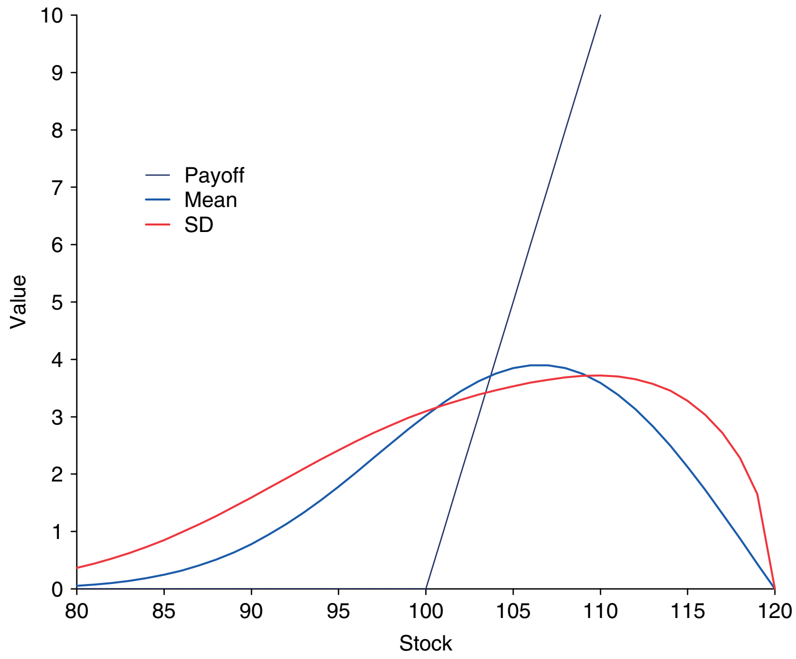

Figure 11: Mean and standard deviation for the optimally hedged up-and-out call barrier option. See text for parameters.

We then find a $(q_1, \dots, q_6)$ that maximizes

$$m^{(1)}(S, \sigma, 0) - \sum_{i=1}^{6} p(q_i) - \xi \sqrt{v^{(1)}(S, \sigma, 0)}$$

where $p(q_i)$ is the market price of trading q_i shares of strike E_i, allowing for bid-offer prices. Again, in our example we use $\xi = 0.5$.

14.2 Diffusion-eliminating Dynamic Hedge, Optimal Static Hedge

With the diffusion-eliminating delta hedge we find that when $S = 100$ our optimal choice for the static hedge' is given by:

Option	1	2	3	4	5	6
Strike	80	90	100	110	120	130
Quantity	0.14	0.31	0.00	−0.04	−1.25	1.02

The cost of this static hedge is -6.42, the mean, m, is $+3.26$ and the variance, v, is 11.03. The barrier option is therefore valued at 4.82. This is in contrast to the 5.68 value without any static hedge. Recall that we are trying to reduce this value as much as possible so that we can sell the barrier option as cheaply as possible. Results are shown in Figure 12.

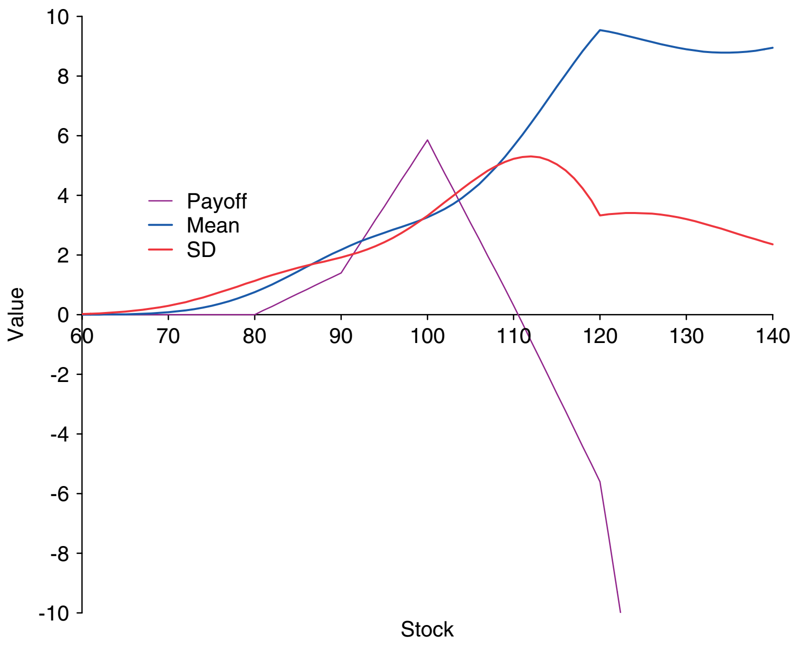

Figure 12: Mean, standard deviation and payoff for the diffusion-eliminating hedged up-and-out call barrier option with optimal static hedge. See text for parameters.

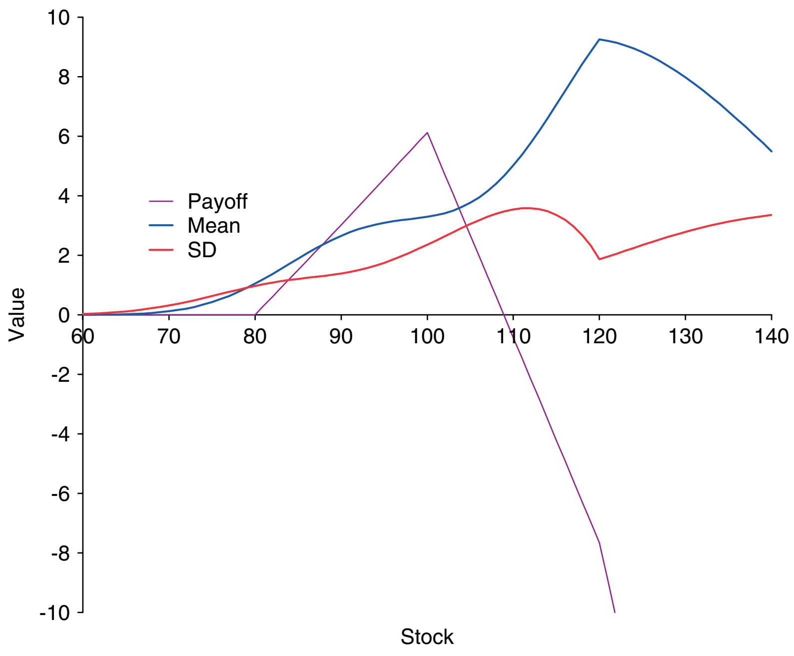

Figure 13: Mean, standard deviation and payoff for the optimally dynamically hedged up-and-out call barrier option with optimal static hedge. See text for parameters.

14.3 Optimal Dynamic Hedge, Optimal Static Hedge

With the optimal delta hedge we find that when $S = 100$ our optimal choice for the static hedge is given by

Option	1	2	3	4	5	6
Strike	80	90	100	110	120	130
Quantity	0.30	0.01	0.00	–0.00	–0.62	0.00

The cost of this static hedge is -6.39, the mean, m, is $+3.29$ and the variance, v, is 5.55. The barrier option is therefore valued at 4.28. This is in contrast to the 4.57 value without any static hedge. Also the variance is dramatically reduced from the suboptimal, non-statically hedged case. Results are shown in Figure 13.

All of the above results are summarized in the following table.

From these examples we can see that the final one, **ODH, OSH** is optimal (as you would expect), both in terms of having the lowest price and is therefore the easiest to sell, and by far the smallest variance, and therefore the smallest jump risk.

	Mean *(m)*	Var. *(v)*	Static Hedge	Value
DEDH, NSH	–3.34	21.85		–5.68
ODH, NSH	–3.02	9.58		–4.57
DEDH, OSH	3.26	11.03	6.423	–4.82
ODH, OSH	3.29	5.55	6.390	–4.28

From all of these examples we can see how we can control to some extent the mean and variance of the barrier option by different forms of dynamic hedging, and also how static hedging can be used to further reduce risk and make a contract's value more appealing. These final results are only possible because the model is non linear.

15 Some thoughts

We have taken the classical jump-diffusion model of Merton and classical mean-variance analysis and put the two together. It is clear from the results that there is far more to jump-diffusion pricing that simple Black–Scholes-type formulæ and naive calibration to vanilla options. Indeed the role of vanilla options ought to be through static hedging rather than calibration since the informational content in vanillas (in terms of volatilities and jump parameters) may be minimal because of supply and demand. It is also clear that adoption of diffusion-eliminating dynamic hedging is inferior to the obvious approach of variance minimization, especially when the justification for its use is on the flimsy, economic, grounds of diversification.

However, rather than pull the rug from under the Merton approach completely we would like to suggest one justification for diffusion-only hedging. The justification is that choosing $\Delta = m_S$ is the only way to eliminate the dependence of m on the difficult-to-estimate parameter μ. This parameter is rarely measured by quants since it does not appear in the ubiquitous complete-market models. Although most models of incomplete markets result in the appearance of μ it still makes quants uncomfortable (and often you will see in the research papers that this parameter morphs into r for no particular reason). So choose $\Delta = m_S$ and it disappears (although it remains in the v equation). We would like to suggest therefore that either Merton was wrong or, if correct, then correct for the wrong reason.

16 Sensitivity to μ

As the final test for this up-and-out call option we calculate the value of the option in all four cases, diffusion-eliminating dynamic hedge or optimal dynamic hedge, and with and without optimal hedge while varying the drift parameter μ. Note that the optimal static hedge has been found assuming that $\mu = 0$, therefore the following results show sensitivity to the parameter μ after it has been estimated at zero. The results are shown in Figure 14. Remember that we want the value to be as high as possible. (Since it is negative we want its absolute value to be as small as possible, we are selling the option.)

For most values of μ the doubly optimal solution is the best. For smaller values of μ the optimal dynamic hedge does better, but had we known μ our static hedge would have been different. For all values of μ in this example the classical diffusion-eliminating hedge without optimal static hedging does the worst.[4]

17 Example 2: Valuing and Hedging a Down-and-out Put, No Static Hedge

We are now going to look at the pricing and hedging of a short down-and-out put option with the same parameter and variable values as in the up-and-out call example, except that the barrier is now at 80.

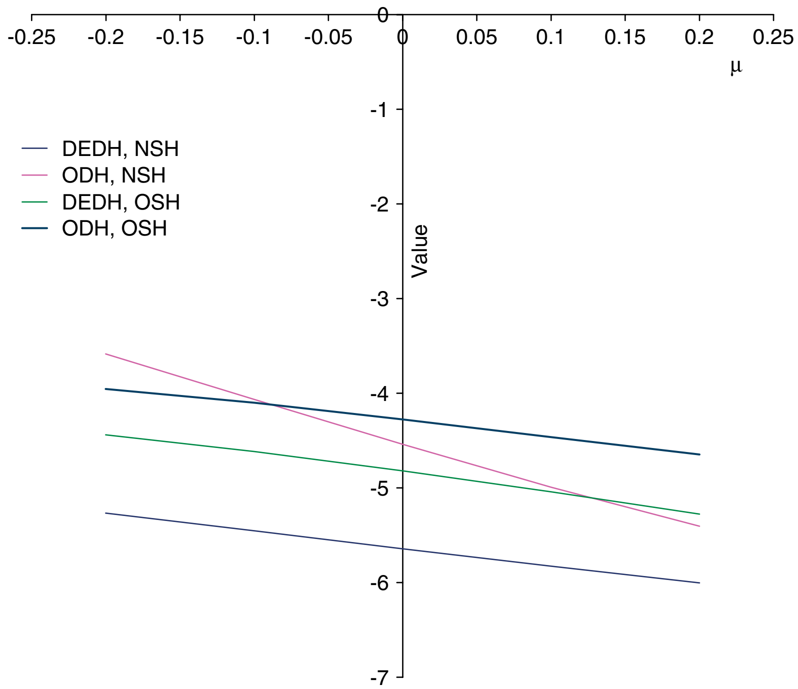

Figure 14: Sensitivity of final value in all four cases.

17.1 Diffusion-eliminating Dynamic Hedge, No Static Hedge

When delta is chosen to eliminate diffusion in the portfolio's value we find that the mean, m, when the underlying has value 100, is −2.52. The variance, v, is 28.24, so that the standard deviation is 5.31. If we were selling this option then we would sell it for

$$-2.52 - \xi \times 5.31.$$

With $\xi = 0.5$ we would sell the option for 5.17 and no less.

17.2 Optimal Dynamic Hedge, No Static Hedge

When delta is chosen optimally we find

$$m = -3.00, \quad \text{and} \quad v = 13.24$$

so we would sell for 4.82.

See Figures 15 and 16 for means and standard deviations against the underlying asset for these two cases.

18 Example 2 continued: Valuing and Hedging a Down-and-out Put, Static Hedge

Now let us introduce into our universe the same six call options as before.

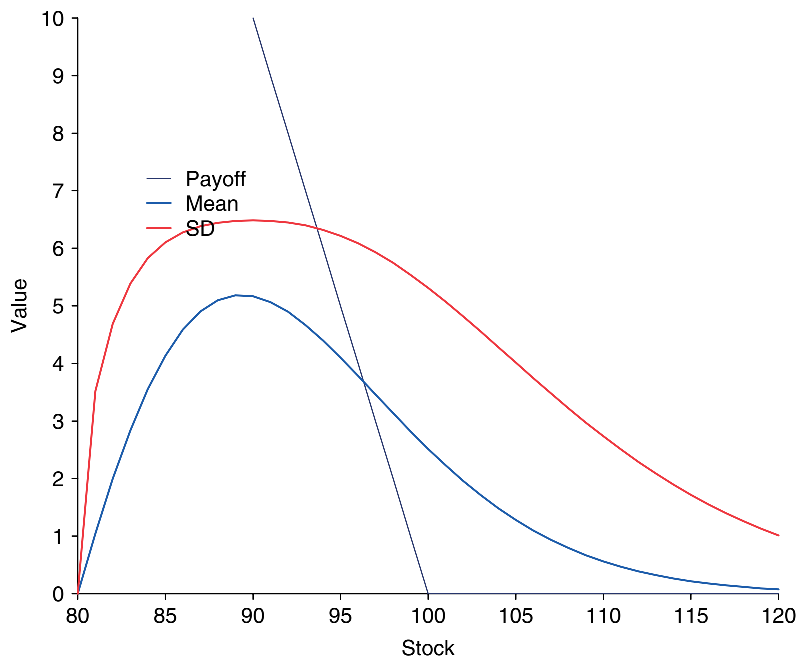

Figure 15: Mean and standard deviation for the diffusion-eliminating hedged down-and-out put barrier option. See text for parameters.

18.1 Diffusion-eliminating Dynamic Hedge, Optimal Static Hedge

With the diffusion-eliminating delta hedge we find that when $S = 100$ our optimal choice for the static hedge' is given by:

Option	1	2	3	4	5	6
Strike	80	90	100	110	120	130
Quantity	−0.44	−0.30	0.51	0.01	0.00	0.58

The cost of this static hedge is -10.09, the mean, m, is 13.25 and the variance, v, is 7.15. The barrier option is therefore valued at 4.49. This is in contrast to the 5.17 value without any static hedge. Results are shown in Figure 17.

18.2 Optimal Dynamic Hedge, Optimal Static Hedge

With the optimal delta hedge we find that when $S = 100$ our optimal choice for the static hedge is given by:

Option	1	2	3	4	5	6
Strike	80	90	100	110	120	130
Quantity	−0.57	−0.00	0.15	0.01	0.05	0.59

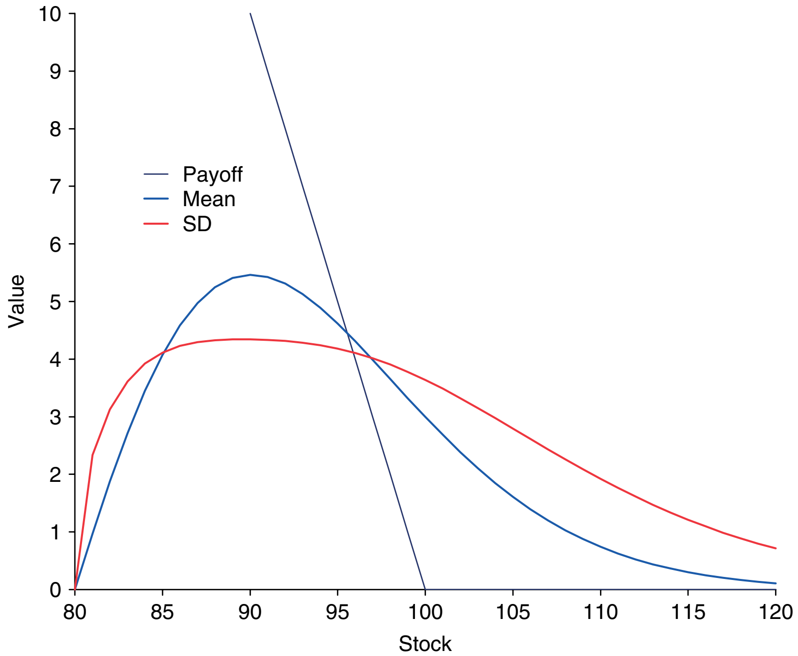

Figure 16: Mean and standard deviation for the optimally hedged down-and-out put barrier option. See text for parameters.

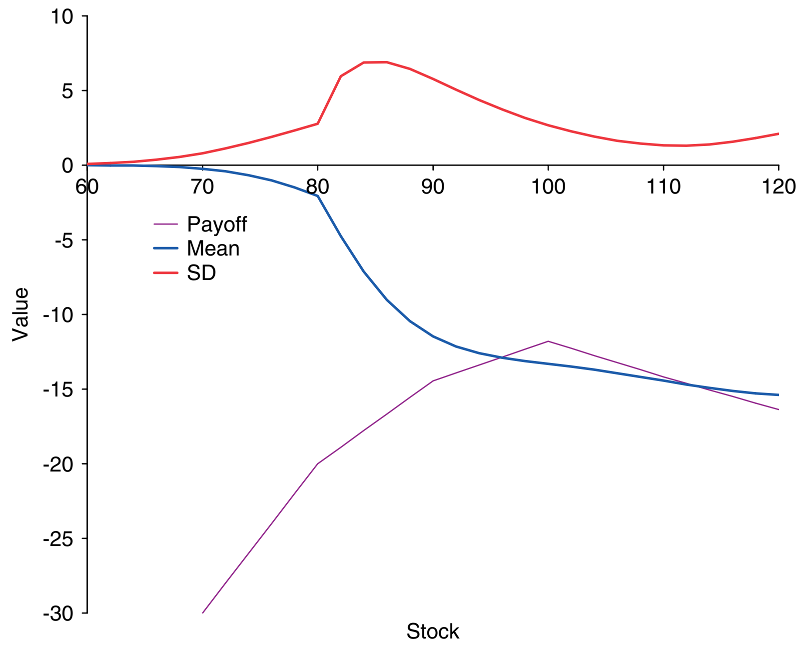

Figure 17: Mean, standard deviation and payoff for the diffusion-eliminating hedged down-and-out put barrier option with optimal static hedge. See text for parameters.

The cost of this static hedge is -11.16, the mean, m, is 14.18 and the variance, v, is 4.32. The barrier option is therefore valued at 4.05. This is in contrast to the 4.82 value without any static hedge. Results are shown in Figure 18.

All of the above results are summarized in the following table.

	Mean *(m)*	Var. *(v)*	Static Hedge	Value
DEDH, NSH	−2.52	28.24		−5.17
ODH, NSH	−3.00	13.24		−4.82
DEDH, OSH	−13.25	7.15	−10.09	−4.49
ODH, OSH	−14.18	4.32	−11.16	−4.05

In terms of both value and variance reduction the down-and-out put is even more impressive than the up-and-out call.

19 Other definitions of 'value'

In the above examples we have statically hedged so as to find the best value according to our definition of value. This is by no means the only static hedging strategy. One can readily imagine different players having different criteria.

Obvious strategies that spring to mind are as follows.

- Minimize variance, that is minimize the function v. This has the effect of reducing model risk as much as possible using all available instruments (the underlying and all traded options). This may be a strategy adopted by the sell side.

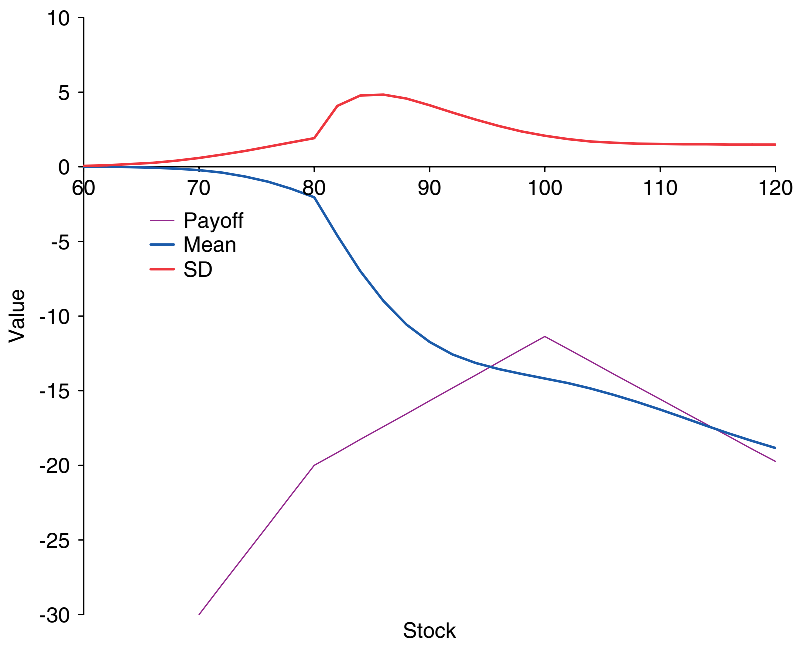

Figure 18: Mean, standard deviation and payoff for the optimally dynamically hedged down-and-out put barrier option with optimal static hedge. See text for parameters.

- Maximize the return-risk ratio. This is perhaps more of a buy-side strategy, for maximizing Sharpe ratio, for example.

Appendix: Application to Lévy processes

According to the Lévy-Khinchin representation, a Lévy process X is essentially a sum of Wiener processes, Poisson processes, and a straight line:

$$\frac{1}{t}\log E[e^{i\theta X(t)}] = i\gamma\theta - \frac{1}{2}\sigma^2\theta^2 + \int e^{i\theta x} - 1 - i\theta x 1(|x| < 1)\nu(dx), \tag{A.1}$$

where γ is a coefficient for the straight line, σ is the scale of the Wiener process, and ν is a positive measure which satisfies:

$$\int_{-1}^{1} x^2\nu(dx) + \int_{|x|>1}\nu(dx) < \infty.$$

The Lévy measure ν aggregates many Poisson jumps in such a way that small jumps can occur more often, while large jumps are sparse, to satisfy the regularity condition for a stochastic process: applying Taylor expansion of the integrand in (A.1), one can see that the integrability condition above indeed makes the integral in (A.1) valid. Many well known Lévy processes, such as alpha-stable processes, have infinitely many jumps in a finite time period with probability one, and also provide closed-form transition probability density functions. From a practical point of view, the tiny jumps are not visible due to the fluctuation of the Wiener process, at least numerically, and we might as well omit them and make the measure finite:

$$\int \nu(dx) = \lambda.$$

Especially when we are dealing with complicated path integrals to determine the value of an exotic payoff, there is no loss by doing so. In this case, the methodology in this article can be exploited by setting the distribution of the jumps as follows:

$$E[f(J)] = \frac{1}{\lambda}\int f(e^x)\nu(dx).$$

In other words, the scaled Lévy measure is the probability distribution of the logarithm of the jump. An additional contribution to the straight line in this case is

$$-i\theta\int_{|x|<1} x\nu(dx).$$

FOOTNOTES & REFERENCES

1. This is a quantitative finance joke.
2. In the jump-to-zero case we could have taken the traditional route of hedging one option with another to eliminate jump risk. This introduces the market price of jumps risk. Again traditionally, this would then be determined by calibration. Unfortunately, calibrated market prices of risk tend to be highly unstable, and certainly more unstable than historical parameters. The model here does not rely on the existence or stability of the market price of risk, but rather relies on the relative stability of the historical, real, parameters.
3. And even to Levy processes, as discussed in the appendix.
4. Note that this test assumes that μ is constant, and then varies it. So the sensitivity to drift may be greater than here if we were to use a more general, weaker, assumption for the drift.

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