THE MATHEMATICS OF GAMBLING

By
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About the Author

Edward Thorp is adjunct professor of finance and mathematics at the University of California at Irvine, where he has taught courses in finance, probability and functional analysis. He previously taught at the University of California at Los Angeles, the Massachusetts Institute of Technology and New Mexico State University.

Thorp's interest in gambling dates back almost 30 years, while he was still in graduate school at UCLA. It was here that he first formulated his dream of making money from the development of a scientifically-based winning gambling system. His first subject of study was the roulette wheel, which offered him the opportunity to use modern physics to predict the resting place of the ball.

With the roulette work unfinished, Thorp's attention was diverted by the blackjack work of Baldwin, Cantey, Maisel and McDermott. He set to work on this new problem. With the aid of a computer, Thorp developed the basic strategy and the five-count, ten-count and ultimate counting strategies. He used these methods with success in the Nevada casinos. The work was first publicized in a scientific journal and saw broad public exposure in the 1962 book *Beat the Dealer*. The book underwent a revision in 1966 and it is still regarded as the classic early work in the "blackjack revolution" which continues to this day.

In the late 1960s, Thorp developed with Sheen Kassouf a successful method for stock market investing involving warrants that proved so profitable that Thorp turned $40,000 into $100,000 in two years. The strategy was published in *Beat the Market* in 1967. Additionally, using this strategy and further refinements, Thorp manages a multi-million-dollar investment portfolio. He is President of Oakley Sutton Management Corp. and Chairman of the the Board of Oakley Sutton Securities Corp.

Thorp has continued to advance new theories for gambling and other games, as well as the stock market.

Section One

Card Games

Casino card games such as baccarat and blackjack differ significantly from casino games such as craps, roulette, and slot machines in that they are not independent trial processes-that is, the cards that already have been played do affect the odds on subsequent hands.

Consider for a moment the game of blackjack, where the cards used on a round are put aside and successive rounds are dealt from an increasingly depleted pack. The cards are reshuffled before a round if the remaining unused cards would be insufficient to complete a round or earlier, usually at the casino's discretion. What the early research on blackjack (contained in *Beat the Dealer*) showed and what has been confirmed repeatedly in the intervening 23 years is that the end pack provides favorable situations often enough to give the player an overall advantage.

While it is foolish to keep a record of past decisions at craps in order to determine which numbers are "hot" or "cold" (the dice have no memory), an ability to keep track of which cards have been played and knowledge of their relationship to the player's expectation can be beneficial, as long as the cards are not reshuffled after every hand.

The ability to keep track of the cards played does not alone guarantee gambling success at a particular game. Indeed, one of the chief tasks of this section will be to examine the usefulness of card counting strategies in baccarat, considering the bets offered and the nature of the game.

In Chapter 2, we will comment on blackjack systems, as well as statistical methods useful in detecting casino cheating. The latter subject is important to those who play the game seriously, because cheating incidents can erode any small edge the player may gain through the use of basic strategy and card counting.

Chapter 1

Introductory Statement

The casino patron who decides to "try his luck" at the tables and the horse player who wagers at the racetrack confront what seem to be formidable adversaries. The casinos hope to have the advantage on every bet offered and, at the track, the pari-mutuel takeout of 17-25% on every bet assures all but a few will wind up losers.

As soon as he enters the casino, the player must make several important decisions, the first being: What game do I play? Even after this choice is made, most games offer additional options: Do I play individual numbers or the even-money bets in roulette? Do I stand with a pair of eights in blackjack or should I hit or split the pair? Should I bet pass or the one-roll propositions in craps?

The horse player is offered a number of choices as well. He is usually faced with a field of six to 12 horses. He can play one or more horses to win, place, or show, in addition to combining any number of horses in the exotic or "gimmick" wagers.

All of these choices have "right" answers, if the player seeks to maximize his return or minimize his loss. They all can be at least partially solved through the use of mathematical theory. The intelligent player must have a basic understanding of the mathematics behind the game or games he plays if he is to survive financially or actually profit. There *are* situations where the player has the advantage. The most-publicized example, of course, is casino blackjack. The game has become tougher in recent years due to casino countermeasures, but blackjack can still be profitable for the sophisticated player. There could be several other favorable games, as the reader will soon discover.

A familiarity with basic probability will allow the alert gambler to discover those positive expectancy games and exploit them where they exist. A vast knowledge of mathematics is not required. Some of the finest poker players in the country never went to college, but they do have a sense of what makes a good poker hand and what their chances of having the best hand are after all the cards have been dealt.

Mathematical Expectation

I have already made reference to the concept of mathematical expectation. This principle is central to an understanding of the chapters to follow.

Imagine for a moment a coin toss game with an unbiased coin (a coin we assume will produce 50% heads and 50% tails). Suppose also that we are offered an opportunity to bet that the next flip will be heads and the payoff will be even money when we win (we receive a $1 profit in addition to the return of wager). Our mathematical expectation in this example is:

$$(.5)(1) + (.5)(-1) = 0$$

The *mathematical expectation* of any bet in any game is computed by multiplying each possible gain or loss by the probability of that gain or loss, then adding the two figures. In the preceding example, we expect to gain nothing from playing this game. This is known as a *fair game*, one in which the player has no advantage or disadvantage.

Now suppose the payoff was changed to 3/2 (a gain of $1.50 in addition to our $1 bet). Our expectation would change to:

$$(.5)(1.50) + (.5)(-1) = +.25$$

Playing this game 100 times would give us a positive expectation of $25.

The two examples presented thus far are admittedly simple, but often this type of analysis is all that is needed to evaluate a proposition. Consider the "dozens" bet in roulette. Our expectation for a $1 bet is:

$$(12/38)(2) + (26/38)(-1) = -.0526$$

As another example, suppose that on the first hand of four-deck blackjack the player bets $12, he is dealt 6,5, and the dealer then shows an ace up. The dealer asks the player if he wants insurance. This is a separate $6 bet. It pays $12 if the dealer's hole card is a ten-value. It pays −$6 otherwise. A full four-deck pack has 64 tens and 144 non-tens. Assuming the deck is "randomly" shuffled (this means that all orderings of the cards are equally probable), the chances are equally likely that each of the 205 unseen cards is the dealer's hole card. Thus the player's expectation is:

$$(64/205)(12) + (141/205)(-6) = -78/205$$

or about −$.38. The player should not take insurance.

Different betting amounts have different expectations. But the player's expectation as a *percent of the amount bet* is always the same number. In the case of betting on the Red in roulette, this is 18/38 −20/38 = −2/38 = −1/19 or about −5.26%. Thus, the expectation of any size bet on Red at American double-zero roulette is −1/19 or about −5.26% of the total amount bet. So to get the expectation for any size bet on Red, just multiply by −5.26%. With one exception, the other American double-zero roulette bets also have this expectation per unit bet. The player's expectation per unit is often simply called the player's disadvantage. What the player loses, the house wins, so the *house advantage, house percentage,* or house expectation per unit bet by the player is +5.26%.

A useful basic fact about the player's expectation is this: the expectation for a series of bets is the total of the expectations for the individual bets. For instance, if you bet $1 on Red, then $2, then $4, your expectations are −$2/38, −$4/38, and −$8/38. Your total expectation is −$14/38 or (a loss of) about −$.37. Thus, if your expectation on each of a series of bets is −5.26% of the amount bet, then the expectation on the whole series is −5.26% of the total of all bets. This is one of the fundamental reasons why "staking systems" don't work: a series of negative expectation bets *must* have negative expectation.

Repeated Trials

Expectation is the amount you *tend* to gain or lose on average when you bet. It, however, does not explain the fluctuations from expectation that occur in actual trials.

Consider the fair game example mentioned earlier in the chapter. In a series of any length, we have an expectation of 0. In any such series it is possible to be ahead or behind. Your total profit or loss can be shown to have an average deviation from expectation of about $\sqrt{N}$. Let D = T −E be the difference of deviation between what you actually gain or lose (T), and the expected gain or loss (E). Therefore, for 100 bets, the average deviation from E = 0 is about $10 (in fact, the chances are about 68% that you'll be within $10 of even; they're about 96% that you'll be within $20 of even). For ten thousand $1 bets it's about $100 and for a million $1 bets it's about $1,000. Table 2-1 shows what happens. For instance, the last line of Table 1-1 says that if we match coins one million times at $1 per bet, our expected gain or loss is zero (a "fair" game). But on average, we'll be about $1,000 ahead or behind. In fact, we'll be between +$1,000 and −$1,000 about 68% of the time. (For a million $1 bets, the deviation D has approximately a normal probability distribution with mean zero and standard deviation $1,000.) We call the total of the bets in a series the "action," A. For one series of one million $1 bets, the action is $1,000,000. However (fifth column) D/A=0.001, so

Table 1-1

number of bets N (A = $N)	expected gain E	average size of D is about $\sqrt{N}$	about 68% of time T is between	average size of D/A	about 68% of time T/A is between
100	0	10	10 and - 10	0.1	0.1 and -0.1
10,000	0	100	100 and - 100	0.01	0.01 and -0.01
1,000,000	0	1000	1000 and -1000	0.001	0.001 and -0.001

the deviation as a percent of the action is very small. And about 68% of the time T/A is between −.001 and +.001 so as a percent of the action the result is very near the expected result of zero. Note that the average size of D, the deviation from the expected result E, grows—contrary to popular belief. However, the average size of the percentage of deviation, D/A, tends to zero, in agreement with a correct version of the "law of averages."

For $1 bets on Red at American roulette, the corresponding results appear in Table 1-2. Notice that in the last column the spread in T/A gets closer and closer to E/A = −.0526. This is where we get the statement that if you play a "long time" you'll lose about 5.26% *of the total action*. Note, too, in column 4 that there appears to be less and less chance of being ahead as the number of trials goes on. In fact, it can be shown that in all negative expectation games the chance of being ahead tends to zero as play continues.

Using the concept of *action*, we can now understand the famous "law of averages." This says, roughly, that if you make a *long* series of bets and record both the action (A) and your total profit or loss (T), then the *fraction* T/A is approximately the same as the *fraction* E/A where E is the total of the *expected* gain or loss for each bet. Many people misunderstand this "law." They think that it says the E and T are approximately the same after a long series of bets. This is false. In fact, the difference between E and T tends to get larger as A gets bigger.

Now, the ordinary player probably won't make a million $1 bets. But the casino probably will see that many and more. From the casino's point of view, it doesn't matter whether one player makes all the bets or whether a series of players does. In either case, its profit in the long run is assured and will be very close to 5.26% of the action. With many players, each making some of the 1,000,000 bets, some may be lucky and win, but these will generally be compensated for by others who lose more than the expected amount. For instance, if each of 10,000 players take turns making a hundred $1 bets, Table 1-2 tells us that about 68% of the time their result will be between +$4.74 and −$15.26.

Table 1-2

N (A = $N)	E	D ~ $\sqrt{N}$	about 68% of time T is between	D/A ~ $1/\sqrt{N}$	about 68% of time T/A is between
100	- $5.26	10	+ $4.74 and - $15.26	0.1	+.0474 and -.1526
10,000	- $526.00	100	- $426.00 and - $626.00	0.01	-.0426 and -.0626
1,000,000	-$52631.00	1,000	-$51631.00 and -$53631.00	0.001	-.0516 and -.0536

About 16% of the time the player wins more than \$4.74 ("lucky") and about 16% of the time the player loses more than \$15.26 ("unlucky"). But players cannot predict or control which group they'll be in.

This same "law of averages" applies to more complicated sequences of bets. For instance, suppose you bet \$10 on Red at roulette (E = −.53), then bet \$100 on "players" at Baccarat (E = −\$1.06), then bet \$10 on a hand in a single-deck blackjack game where the ten-count is 15 tens, 15 others (E = +\$.90). The total E is \$.53 − \$1.06 + \$.90 = −\$.69. The total A is \$10 + \$100 + \$10 = \$120. If you make a long series of bets and record E and A as well as your gains and losses for each one, then just as in the coin matching example (Table 1-1) and the roulette example (Table 1-2), the fraction D/A tends to zero so T/A tends to E/A. That means that over, say, a lifetime, your total losses as a *percent* of your total action will tend to be very close to your total expectation as a percent of your action.

If you want a good gambling life, make positive expectation bets. You can, as a first approximation, think of each negative expectation bet as charging your account with a tax in the amount of the expectation. Conversely, each positive expectation bet might be thought of as crediting your account with a profit in the amount of the expectation. If you only pay tax, you go broke. If you only collect credits, you get rich.

Chapter 2

Blackjack

Blackjack, or twenty-one, is a card game played throughout the world. The casinos in the United States currently realize an annual net profit of roughly one billion dollars from the game. Taking a price/earnings ratio of 15 as typical for present day common stocks, the United States blackjack operation might be compared to a \$15 billion corporation.

To begin the game a dealer randomly shuffles the cards and players place their bets. The number of decks does not materially affect our discussion. It generally is one, two, four, six or eight. There are a maximum and minimum allowed bet.

The players' hands are dealt after they have placed their bets. Each player then uses skill in his choice of a strategy for improving his hand. Finally, the dealer plays out his hand according to a fixed strategy which does not allow skill, and bets are settled. In the case where play begins from one complete randomly shuffled deck, an approximate best strategy (i.e., one giving greatest expected return) was first given in 1956 by Baldwin, Cantey, Maisel, and McDermott.

Though the rules of blackjack vary slightly, the player following the Baldwin group strategy typically has the tiny edge of +.10%. (The pessimistic figure of −.62% cited in the Baldwin's group's work was erroneous and may have discouraged the authors from further analysis.) These mathematical results were in sharp contrast to the earlier and very different intuitive strategies generally recommended by card experts, and the associated player disadvantage of two or three percent. We call the best strategy against a complete deck the *basic strategy*. Determined in 1965, it is almost identical with the Baldwin group strategy and it gives the player an edge of +.13 against one deck and −.53 against four decks.

If the game were always dealt from a complete shuffled deck, we would have repeated independent trials. But for compelling practical reasons, the deck is not generally reshuffled after each round of play. Thus as successive rounds are played from a given deck, we have sampling without replacement and dependent trials. It is necessary to show the players most or all of the cards used on a given round of play before they place their bets for the next round. Knowing that certain cards are missing from the pack, the player can, in principle, repeatedly recalculate his optimal strategy and his corresponding expectation. (The strategies for various card counting procedures, and their expectations, were determined directly from probability theory with the aid of computers. The results were reverified by independent Monte Carlo calculations.)

Blackjack Systems

All practical winning strategies for the casino blackjack player, beginning with my original work in 1961, are based on this knowledge of the changing composition of the deck. In practice each card is assigned a point value as it is seen. By convention the point value is chosen to be positive if having the card out of the pack significantly favors the player and negative if it significantly favors the casino. The magnitude of the point value reflects the magnitude of the card's effect but is generally chosen to be a small integer for practical purposes. Then the cumulative point count is taken to be proportional to the player's expectation.

To a surprising degree, the player's best strategy and corresponding expectation depend only on the fractions of each type of card currently in the pack and only change slowly with the size of the pack. Thus the better systems "normalize" by dividing the cumulative point count by the total number of as yet unseen cards. Most point count systems are initialized at zero cumulative total for the full pack, and the normalized cumulative count is taken to indicate the change in player expectation from the value for the full pack.

The original point count systems, the prototypes for the many subsequent ones, were my five count, ten count, and "ultimate strategy." An enormous amount of effort by many investigators has since been expended to improve upon these count systems. Some of these systems are shown in Table 2-1 (courtesy of Julian Braun).

The idea behind these point count systems is to assign point values to each card which are proportional to the observed effects of deleting a "small quantity" of that card. Table 2-2 (courtesy of Julian Braun, private correspondence) shows this for one deck and for four decks, under typical Las Vegas rules. One must compromise between simplicity (small integer values) and accuracy. My "ultimate strategy" is a point count based on moderate integer values which fits quite closely the data available in the early 1960s. Until recently all the other count systems were simplifications of the "ultimate."

System 1 (Table 2-1) does not normalize by the number of remaining cards. Thus the player need only compute and store the cumulative point count. Normalization gives the improved results of system 2, but requires the added effort of computing the number of remaining cards and of dividing the point count by the number of remaining cards when decisions are to be made. In practice the player can estimate the unplayed cards by eye and use it with system 1 and get almost the results of system 2 with much less effort. Systems 2, 3, 5 and 7 all divide by the number of remaining cards.

Table 2-1

Table 2-1, Braun's simulation of various point count systems. "Bet 1 to 4" means that 1 unit was bet except for the most advantageous x% of the situations, when 4 was bet. To compare systems, x was approximately the same in each case, 21(%).

	RESULTS OF SIMULATED DEALS-PLAYER'S ADVAN.	
STRATEGY/SYSTEM	Flat Bet	Bet 1 to 4
1 Basic Braun + −	.2%	1.4%
2 Braun + −	.7%	2.0%
3 Revere Pt. Ct.	.6%	2.1%
4 Revere Adv. + −	.5%	1.6% to 1.8% +
5 Revere Adv. Pt. Ct. − 71	.6%	2.0%
6 Revere Adv. Pt. Ct. − 73	.8%	2.1% to 2.3% +
7 Thorp Ten Count	.7%	1.9%
8 Hi-Opt	.8%	2.1% to 2.3% +

Table 2-1 (continued)

SYSTEM #	BASIC POINT COUNTS CARD 2	3	4	5	6	7	8	9	T	A	+N?†	SIDE ACE COUNT?	λ(C) 1 DECK	λ(C) 4 DECK
1	1	1	1	1	1	0	0	0	−1	−1	NO	NO	.963	.961
2	1	1	1	1	1	0	0	0	−1	−1	YES	NO	.963	.961
3	1	2	2	2	2	1	0	0	−2	−2	YES	NO	.979	.977
4	1	1	1	1	1	0	0	−1	−1	0	YES	YES	.885 (.968)*	.884 (.968)*
5	2	3	3	4	3	2	0	−1	−3	−4	YES	NO	.994	.994
6	2	2	3	4	2	1	0	−2	−3	0	YES	YES	.924 (.995)*	.922 (.997)*
7	4	4	4	4	4	4	4	4	−9	−4	YES	NO	.720	.709
8	0	1	1	1	1	0	0	0	−1	0	YES	YES	.874 (.938)*	.871 (.939)*

† Normalized.
* Improvement with ace adjustment.

Table 2-2
Changes in Player Expectation by Removing Individual Cards

One deck Top of Deck Expectancy = 0.10%											
Cards removed	A	2	3	4	5	6	7	8	9	10	d*	d/13
Expectation (%)	−.48	.50	.56	.70	.86	.56	.41	.12	−.06	−.39		
Change in Expectation (%)	−.58	.40	.46	.60	.76	.46	.31	.02	−.16	−.49	.31	.024
u_i^*	−.604	.376	.436	.576	.736	.436	.286	−.004	−.184	−.514		
Four decks Top of Deck Expectancy = −0.532%												
Cards removed	A	2	3	4	5	6	7	8	9	10	d*	d/13
Expectation (%)	−1.130	−.147	−.081	.059	.236	−.078	−.239	−.525	−.714	−1.019		
Change in Expectation (%)	−.598	.385	.451	.591	.768	.454	.293	.007	−.182	−.487	.221	.017
u_i^*	−.615	.368	.434	.574	.751	.437	.276	−.010	−.199	−.504		

*See Appendix A.

Systems 4, 6 and 8, which are also normalized, have the first new idea. They assign a point count of zero to the ace for strategy purposes. This is consistent with the evidence: in most instances that have been examined, the optimal strategy seems to be relatively unaffected by changes in the fraction of aces in the pack. However, the player's expectation is generally affected by aces more than by any other card (Table 2-2). Therefore these systems keep a separate ace count. Then the deviation of the fraction of aces from the normal 1/13 is incorporated for calculating the player's expectation for betting purposes.*

The (c) column in Table 2-1 still remains to be explained. It is a numerical assessment of a particular system's closeness to an ideal system based on the change in expectation values contained in Table 2-2. The calculation of the (c) value eliminates the necessity of simulating a large number of hands (say a million) to evaluate a strategy. The computation of these numbers requires some advanced mathematical background, so its explanation is left to the appendix.

Cheating: Dealing Seconds

Various card counting systems give the blackjack player an advantage, provided that the cards are well shuffled and that the game is honest. But many methods may be used to cheat the player. I have been victimized by most of the more common techniques and have catalogued them in *Beat the Dealer.*

One of the simplest and most effective ways for a dealer to cheat is to peek at the top card and then deal either that card or the one under it, called the second. A good peek can be invisible to the player. A good second deal, though visible to the player, can be done so quickly and smoothly that the eye generally will not detect it. Although the deal of the second card may sound different from the deal of the first one, the background noise of the casinos usually covers this completely. Peeking and second dealing leave no evidence. Because these methods are widespread, it is worth knowing how powerful they are.

Does even a top professional blackjack counter have a chance

*See Appendix A, pg. 131.

against a dealer who peeks and deals seconds? Consider first the simple case of one player versus a dealer with one deck. This is an extreme example, but it will illustrate the important ideas.

I shuffle the deck and hold it face up in order to deal practice hands. Because I can see the top card at all times, dealing from a face-up deck is equivalent to peeking on each and every card. I will deal either the first or second card, depending on which gives the dealer the greatest chance to win. I will think out loud as an imaginary dealer might, and the principles I use will be listed as they occur. The results for a pass through one deck are listed in Table 2–3 (pp. 20-21). There were nine hands and the dealer won them all.

On hands one, two, four, six, eight and nine, the dealer wins by busting the player. Because there is only one player, it does not matter what cards the dealer draws after the player busts.

When there are two or more players, the dealer may choose a different strategy. If, for example, the dealer wishes to beat all the players but doesn't want to peek very often, an efficient approach is simply to peek when he can on each round of cards until he finds a good card for himself on top. He then retains this card by dealing seconds until he comes to his own hand, at which time he deals the top card to himself. That strategy would lead to the dealer having unusually good hands at the expense of the collective player hands; because some good hands have been shifted from the players, the player hands would be somewhat poorer than average.

A player could detect such cheating by tallying the number of good cards (such as aces and 10s) which are dealt to the dealer as his first two cards and comparing that total with the number of aces and 10s predicted by theory. In Peter Griffin's book, *The Theory of Blackjack*, he describes how he became suspicious after losing against consistently good dealer hands. Griffin writes that he "...embarked on a lengthy observation of the frequency of dealer up cards in the casinos I had suffered most in. The result of my sample, that the dealers had 770 aces or 10s out of 1,820 hands played, was a statistically significant indication of some sort of legerdemain." Griffin's tally is overwhelming evidence

that something was peculiar. The odds against such an excess of ten-value cards and aces going to the dealer in a sample this size are about four in ten thousand.

Another approach the dealer might select is to beat one player at the table while giving everybody else normal cards. To do this, the dealer peeks frequently enough to give himself the option of dealing a first or second to the unfortunate player each time that player's turn to draw a card comes up. Dealing stiffs to a player so that he is likely to bust is, as we see from the chart in Table 2-3, so easy to do that the player has little chance.

If all dealers peeked and dealt seconds according to the cheating strategy indicated in Table 2-3, I estimate that with one player versus the dealer, the dealer would generally win at least 95 percent of the time. With one dealer against several players, the dealer would win approximately 90 percent of the time. Anyone who is interested can get a good indication of what the actual numbers are by dealing a large number of hands and recording the results.

The deadliest way a dealer can cheat is to win just a few extra hands an hour from the players. This approach is effective because it is not extreme enough to attract attention, or to be statistically significant and therefore detectable over a normal playing time of a few hours. For example, the odds in blackjack are fairly close to even for either the dealer or the player to win a typical hand. Suppose that by cheating the dealer shifts the advantage not to 100 percent but to just 50 percent in favor of the house. What effect does this have on the game?

If we assume that the player plays 100 hands, a typical total for an hour's playing time, and we also assume that the player bets an average of two units per hand, then being cheated once per 100 hands reduces the player's win by one unit on the average. A professional player varying his bet from one to five units would probably win between five and 15 units per hour. The actual rate would depend upon casino rules, the player's level of skill, and the power and variety of winning methods that he employed. Let's take a typical professional playing under good conditions and

Table 2-3

Hand	Top Card	Card Dealt	Comment	Plr. Gets	Plr. Total	Dlr. Gets	Dlr. Total	Result
1	2	First		2	2			
	5	Second (4)	Dlr. trys for good card			4	4	
	5	Second (10)	Dlr. will have 9; Plr. gets stiff	10	12			
	5	First				5	9	
	4	First	Worsens Plr. stiff	4	16			
	5	Second (8)	Prevent Plr. 21	8	24 bust			Dlr. wins
	5	First	Doesn't matter			5	14	
	9	First	Doesn't matter			9	23 bust	
2	4	First		4	4			
	K	First				10	10	
	Q	First	Give Plr. stiff	10	14			
	J	Second (J)	J will bust Plr. (Second turns out to be J, too!)	J	24 bust			Dlr. wins
	J	First	Doesn't matter			J	20	
3	A	Second (3)		3	3			
	A	First				A	1, 11	
	3	First	Bldg. potential stiff	3	6			
	J	First				J	BJ	Dlr. wins
4	2	First		2	2			
	9	First	Would make Plr. 11, so Dlr. takes			9	9	
	K	First	Give Plr. stiff	K	12			
	Q	Second (3)	Guarantees Dlr. win			3	12	
	Q	First		Q	22 bust			Dlr. wins
	8	First				8	20	
5	5	First		5	5			
	J	First				J	10	
	A	Second (A)	Guarantees Dlr. win (Second was A, too!)	A	6,16			
	A	First				A	BJ	Dlr. wins

Table 2-3 (continued)

Hand	Top Card	Card Dealt	Comment	Plr. Gets	Plr. Total	Dlr. Gets	Dlr. Total	Result
6	8	First		8	8			
	Q	First				Q	10	
	6	First	Give Plr. stiff	6	14			
	2	Second (3)				3	13	
	2	First	Give Plr. worse stiff	2	16			
	6	First		6	22 bust			Dlr. wins
	4	First				4	17	
7	7	First		7	7			
	6	Second (A)				A	1, 11	
	6	First		6	13			
	6	Second (K)				K	BJ	Dlr. wins
8	6	First		6	6			
	10	First				10	10	
	K	First	Give Plr. bad stiff	K	16			
	9	Second (2)	Dlr. must win			2	12	
	9	First		9	25 bust			Dlr. wins
	10	First	Doesn't matter	10		10	22 bust	
9	8	First		8	8			
	Q	First				Q	10	
	7	First		7	15			
	10	Second (5)	Dlr. must win			5	15	
	10	First		10	25 bust			Dlr. wins
	7	First				7	22	

assume that his win rate is ten units per hour and his average bet size is two units. Given those assumptions, being cheated ten times per hour or one-tenth of the time would cancel his advantage. Being cheated more than ten percent of the time would probably turn him into a loser.

Cheating in the real world is probably more effective than in the hypothetical example just cited, because the calculations for that example assume cheating is equally likely for small bets and big bets. In my experience, the bettor is much more likely to be cheated on large bets than on small ones. Therefore, the dealer who cheats with maximum efficiency will wait until a player makes his top bet. Suppose that bet totals five units. If the cheat shifts the odds to 50 percent in favor of the house, the expected loss is 2-1/2 units, and just four cheating efforts per 100 hands will cancel a professional player's advantage. A cheating rate of five or ten hands per 100 will put this player at a severe disadvantage.

We can see from this that a comparatively small amount of cheating applied to the larger hands can have a significant impact on the game's outcome. This gives you an idea of what to look for when you are in the casinos and think that something may be amiss.

Missing Cards: The Short Shoe

I have heard complaints that cards have been missing from the pack in some casino blackjack games. We'll discuss how you might spot this cheating method.

In 1962, I wrote on page 51 of *Beat the Dealer*, "Counting the. . .cards. . .is an invaluable asset in the detection of cheating because a common device is to remove one or more cards from the deck." Lance Humble discusses cheating methods for four-deck games dealt from a shoe in his International Blackjack Club newsletter. He says, "The house can take certain cards such as tens and aces out of the shoe. This is usually done after several rounds have been dealt and after the decks have been shuffled several times. It is done by palming the cards while they are being

shuffled and by hiding them on the dealer's person. The dealer then disposes of the cards when he goes on his break." But cheating this way is not limited to the casino. Players have been known to remove "small" cards from the pack to tilt the edge their way. The casino can spot this simply by taking the pack and counting it; the player usually has to use statistical methods.

In the cheating trade, the method is known as the *short shoe.* Let's say the dealer is dealing from a shoe containing four decks of 52 cards each. In 52 cards, there should be 16 ten-value cards: the tens, jacks, queens and kings. Logically, in four decks of 208 cards, there should be 64 ten-value cards. I'll call all of these "tens" from now on. Casinos rarely remove the aces—even novice players sometimes count these.

Suppose the shift boss or pit boss takes out a total of ten tens; some of each kind, of course, not all kings or queens. The shoe is shortened from 64 tens to 54 tens, and the four decks from 208 cards to 198 cards.

The loss of these ten tens shifts the advantage from the player to the dealer or house. The ratio of others/tens changes from the normal $144/64 = 2.25$ to $144/54 = 2.67$, and this gains a little over one percent for the house. How can you discover the lack of tens without the dealer knowing it?

Here is one method that is used. If you're playing at the blackjack table, sit in the last chair on the dealer's right. Bet a small fixed amount throughout a whole pack of four decks. After the dealer puts the cut card back only, let's say, ten percent of the way into the four shuffled decks and returns the decks into the shoe, then ready yourself to count the cards. Play your hand mechanically, only pretending interest in your good or bad fortunes. What you're interested in finding out is the number of tens in the whole four-deck shoe.

Let's say the shift boss has removed ten tens. (Reports are that they seem to love removing exactly ten from a four-deck shoe.) When the white cut card shows at the face of the shoe, let's say that the running count of tens has reached 52. That means mathematically that if all 64 ten-value cards were in the shoe, then, of the remaining 15 cards behind the cut card, as many as 12 of them would be tens, which mathematically is very unlikely. This is how one detects the missing ten tens because the dealer never shows their faces but just places them face down on top of the stack of discarded cards to his right, which he then proceeds to shuffle face down in the usual manner preparatory to another four-deck shoe session.

Although at first the running count is not easy to keep in a real casino situation, a secondary difficulty is estimating the approximate number of cards left behind the cut card after all the shoe has been dealt. To practice this, take any deck of 52 cards and cut off what you think are ten, 15 or 20 cards, commit yourself to some definite number, and then count the cards to confirm the closeness of your estimate. After a while, you can look at a bunch of cards cut off and come quite close to their actual number.

In summary, count the number of tens seen from the beginning of a freshly shuffled and allegedly complete shoe. When the last card is seen and it is time to reshuffle the shoe, subtract the number of tens seen from the number that are supposed to be in the shoe—64 for a four-deck shoe—to get the number of unseen ten-value cards which should remain. If 54 ten-value cards were seen, there should be ten tens among the unused cards. Then estimate the number of unseen cards. You have to be sure to add to the estimated residual stack any cards which you did not see during the course of play, such as burned cards. Step four is to ask whether the number of unseen ten-value cards is remarkably large for the number of residual cards. If so, consider seriously the possibility that the shoe may be short. For instance, suppose there are 15 unseen cards, ten of which are supposed to be ten-values. A computation shows that the probability that the last 15 cards of a well-shuffled four-deck shoe will have at least ten ten-value cards is 0.003247 or about one chance in 308.

Thus the evidence against the casino on the basis of this one shoe alone is not overwhelming. But if we were to count down the same shoe several times and each time were to find the remaining cards suspiciously ten-rich, then the evidence would become very

strong. Suppose that we counted down the shoe four times and that each time there were exactly 15 unseen cards. Suppose that the number of unseen tens, assuming a full four decks, was nine, 11, ten, and 13 respectively. Then referring to Table 2-4, the probabilities to six decimal places are $H(9) = .014651$ to have nine or more unseen tens, and for at least 11, ten, and 13 respectively, the chances are $H(11) = .000539$, $H(10) = .003247$, and $H(13) = .000005$. These correspond to odds of about 1/68, 1/1,855, 1/308 and 1/200,000 respectively. The odds against all these events happening together is much greater still. In this example, the evidence strongly suggests that up to nine ten-value cards are missing. There can't be more than nine missing, of course, because we saw all but nine on one countdown.

If the casino shuffles after only 104 cards are seen, it is not so easy to tell if ten ten-value cards were removed. A mathematical proof of this is contained in the appendix.*

This discussion should make it clear that the method suggested is generally not able to easily spot the removal of ten-value cards unless the shoe is counted several times or is dealt down close to the end.

One of the interesting ironies of the short shoe method of cheating players is that neither the shift boss nor the pitboss—the latter bringing the decks of cards to the dealer's table—need tell the dealer that his shoe is short. Thus, the dealer doesn't necessarily have to know that he's cheating. After all, he's just dealing. It's an open question how many dealers know that they're dealing from a short shoe.

Reports are that the short shoe is a frequent method that casinos use in cheating at blackjack using more than one deck. The tables with higher minimums (say $25) are more tempting candidates for short shoes than those with the lower minimums.

An experienced card counter can improve the method by counting both tens and non-tens. Then he'll know exactly how many unseen cards there are, as well as unseen tens. Table 2-4 can then be used with greater confidence.

In practice, you don't need to count through a shoe while bet-

*See Appendix B, pg. 136.

Table 2-4

K Number Of Unseen Ten-Value Cards	P(K) Probability Of Exactly K Unseen Tens	H(K) Probability At Least This Many Unseen Tens
0	.003171	1.000000
1	.023413	.996829
2	.078818	.973416
3	.160423	.894598
4	.220732	.734176
5	.217437	.513443
6	.158380	.296006
7	.086431	.137626
8	.036132	.050782
9	.011404	.014651
10	.002707	.003247
11	.000475	.000539
12	.000059	.000065
13	.000005	.000005
14	.000000	.000000
15	.000000	.000000

ting (and thus losing money in the process) to find out that the casino is cheating. If you suspect foul play, count while standing behind the player to the dealer's right.

You might easily catch a short shoe by simply counting all the cards that are used, whether or not you see what they are. Then if the remaining cards, at the reshuffle, are few enough so you can

accurately estimate their number, you can check the total count. For instance, you count 165 cards used and you estimate that 31 ± 3 cards remain. Then there were 196 ± 3 cards rather than the 208 expected, so the shoe is short.

A casino countermeasure is to put back a 4, 5 or 6 for each ace or ten-value card removed. Then the total number of cards remains 208, and the casino gets an even greater advantage than it would from a short shoe.

Cards do get added to the deck, and there's a spooky coincidence to illustrate this. On page 51 of *Beat the Dealer*, I wrote in 1962, "One might wonder at this point whether casinos have also tried adding cards to the deck. I have only seen it done once. It is very risky. Imagine the shock and fury of a player who picks up his hand and sees that not only are both his cards 5s, but they are also both spades." And then 15 years later in 1977, a player in a one-deck game did get a hand with two of the same card—the 5 of spades. Walter Tyminski's casino gaming newsletter, *Rogue et Noir News*, reported on page 3 of the June 15, 1977 issue, "What would you do if the player at your right in a single blackjack game had two 5 of spades? Nicholas Zaika, a bail bondsman from Detroit, had that experience at the Sahara in Las Vegas on May 24 at a $5 minimum table.

"Zaika wasn't in the best of humor because he had reportedly lost $594,000 at other Sahara tables, by far the largest loss he has ever experienced. Zaika had the blackjack supervisor check the cards and there were 53 cards in the deck, the duplicate bein the 5 of spades. . .The gamer has engaged the services of Las Vegas attorney George Grazadei to pursue claims he feels that he has against the casino. . .

"The Sahara denies any wrongdoing and says that it is cooperating fully with the investigation. . . Players aren't likely to introduce an extra 5 because the presence of the extra 5 favors the house and not the player."

Suppose instead of just counting tens used and total cards used, you kept track of how many aces, 2s, 3s, queens, kings, and so on were used. This extra information should give the player a better chance of detecting the short shoe. The ultimate proof would be to count the number of each of the 52 types of cards which have been used. Mathematical readers might wish to investigate effective statistical or other ways of using information for detecting shoes in which the numbers of some of the cards have been changed.

Chapter 3

Baccarat

The games of baccarat and chemin de fer are well known gambling games played for high stakes in several parts of the world. Baccarat is said to be a card game of Italian origin that was introduced into France about 1190 A.D. Two forms of the game developed. One form was called baccarat and the other was called chemin de fer. The most basic difference between these two games is simply that three hands are dealt in baccarat (called baccarat en banque in England) and two hands are dealt in chemin de fer (called baccarat-chemin de fer in England and Nevada).

The cards ace through nine are each worth their face value and the cards ten, jack, queen and king are each worth zero points. A hand is evaluated as the sum modulo ten of its cards, i.e., only the last digit of the total is counted. The object of the game is to be as close to eight or nine as possible with two cards, or as close to nine as possible with at most three cards if one does not have eight or nine on his first two cards. Then the high hand wins.

The games of baccarat and chemin de fer became popular in public casinos all over Europe, as well as in private games, about 1830. At the present time, one or both of these games are well known in London, southern France, the Riviera, Germany and the United States. A form of chemin de fer, which we shall call Nevada baccarat, has been played in a few Nevada casinos since 1958.

The rules, structure and format of the three games have strong similarities. I studied Nevada baccarat with William E. Walden most intensively because the casinos where it is played were readily accessible. Our techniques can be carried over to the other forms of baccarat and chemin de fer.

We were originally motivated by the observation that baccarat and chemin de fer have several points of resemblance to the game of blackjack, or twenty-one. The fact that practical winning strategies for twenty-one have been discovered suggested that there might also be practical winning strategies for baccarat and chemin de fer. In contrast to the situation in twenty-one, we found that there are no current practical winning strategies for the main part of the game, i.e., for the money Banker and Player bets.

Rules and Procedures

To begin the Nevada baccarat game, eight decks of cards are shuffled and a joker is placed face up near the end. The cards are then put into a wooden dealing box called a shoe. The first card is exposed, and its value is noted, face cards being counted as tens. Then this number of cards is discarded, or "burned."

The table has twelve seats, occupied by an assortment of customers and shills. A shill is a house employee who bets money and pretends to be a player in order to attract customers or stimulate play. We refer to them indiscriminately as "players." There are two principal bets, called "Banker" and "Players." Any player may make either of these bets before the beginning of any round of play, or "coup."

To begin the evening's play, two of the players are singled out. One is termed The Banker and the other is termed The Player. The seats are numbered *counter*clockwise from one to twelve. Player

number one is initially The Banker, unless he refuses. In this case the opportunity passes counterclockwise around the table until someone accepts. The Player is generally chosen to be that player, other than The Banker, who has the largest bet on the Player. We have not noticed an occasion when there were no bets on The Player. When we played, there were shills in the game and they generally bet on The Player (except when acting as The Banker, when they generally bet on The Banker).

The Banker retains the shoe and deals as long as the bet "Banker" (which we also refer to as a bet on The Banker) does not lose. When the bet "Players" (which we also refer to as a bet on The Player) wins, the shoe moves to the player on the right. This player now becomes The Banker. If the coup is a tie, the players are allowed to alter their bets in any manner they wish. The same Banker then deals another coup.

To begin a coup, The Banker and The Player are dealt two cards each. As we noted above, the cards ace through nine are each worth their face value and tens and face cards are each worth zero points. Only the last digit in the total is counted.

After The Banker and The Player each receive two cards, the croupier faces their hands. If either two-card total equals 8 or 9 (termed a natural 8 or a natural 9, as the case may be), all bets are settled at once.

If neither The Player nor The Banker have a natural, The Player and The Banker then draw or stand according to the set of rules in Table 3-1.

The high hand wins. If the hands are equal, there is a tie and no money changes hands. Players are then free to change their bets in any desired manner. If the coup being played is complete when the joker is reached, the shoe ends and the cards are reshuffled. Otherwise the coup is first played out to completion. Then the shoe ends and the cards are reshuffled. However, the casino may reshuffle the cards at any time between coups.

Table 3-1

Player having		
0-5	draws a card	
6-7	stands	
8-9	turns cards over	
Banker having	draws when The Player draws	does not draw when The Player draws
0	none, 0-9	
1	none, 0-9	
2	none, 0-9	
3	none, 0-7, 9	8
4	none, 2-7	0, 1, 8, 9
5	none, 4-7	0-3, 8, 9
6	6, 7	none, 0-5, 8, 9
7	stands	stands
8	turns cards over	turns cards over
9	turns cards over	turns cards over

The Main Bets

Two main bets against the house can be made. One can bet on either The Banker or The Player. Winning bets on The Player are paid at even money. Winning bets on The Banker are paid 0.95 of the amount bet. The five percent tax which is imposed on what otherwise would have been an even-money pay-off is called "vigorish." For eight complete decks, the probability that The Banker wins is 0.458597, and the probability of a tie is 0.095156.

The basic idea of the calculation of these numbers is to consider all possible distinct six-card sequences. The outcome for each sequence is computed and the corresponding probability of that

sequence is computed and accumulated in the appropriate register. Numerous short cuts, which simplify and abbreviate the calculation, were taken.

The house advantage (we use advantage as a synonym for mathematical expectation) over The Player is 1.2351 percent. The house advantage over The Banker is 0.458597X5 percent—1.2351 percent or 1.0579 percent, where 2.2930 percent is the effective house tax on The Banker's winnings. If ties are not counted as trials, then the figures for house advantage should be multiplied by 1/0.904844, which give a house advantage per bet that is not a tie, over The Banker of 1.1692 percent and over The Player of 1.3650 percent. The effective house tax on The Banker in this situation is 2.5341 percent.

We attempted to determine whether or not the abnormal compositions of the shoe, which arise as successive coups are dealt, give rise to fluctuations in the expectations of The Banker and The Player bets which are sufficient to overcome the house edge. It turns out that this occasionally happens but the fluctuations are not large enough nor frequent enough to be the basis of a practical winning strategy. This was determined in two ways. First, we varied the quantity of cards of a single numerical value. The results were negative.

We next inquired as to whether, if one were able to analyze the end-deck perfectly (e.g. the player might receive radioed instructions from a computer), there were appreciable player advantages on either bet a significant part of the time. We selected 29 sets of 13 cards each, each set drawn randomly from eight complete decks. There were small positive expectations in only two instances out of 58. Once The Player had a 3.2% edge and once The Banker had a 0.1% edge.

We next proved, by arguments too lengthy and intricate to give here, that the probability distributions describing the conditional expectations of The Banker and The Player spread out as the number of unplayed cards decreases. Thus there are fewer advantageous bets of each type, and they are less advantageous, as the number of unplayed cards increases above 13. The converse oc-

Table 3-2

Bacarrat (Eight Decks)

Subtract one Card of Value	Change In Player Bet	Advantage of: Banker Bet	Relative Point Values (Banker)	An Approximate Point Count
0	−0.002%	+0.002%	2	0
1	−0.004%	+0.004%	4	1
2	−0.005	+0.005	5	1
3	−0.007	+0.007	7	1
4	−0.012	+0.012	11*	2
5	+0.008	−0.008	−8	−1

Table 3-2 (continued)

6	+0.011	−0.011	−11	−2
7	+0.008	−0.008	−8	−1
8	+0.005	−0.005	−5	−1
9	+0.003	−0.003	−3	0

*Arbitrarily reduced from 12 to 11 so points in pack total zero curs as the number of unplayed cards decreased below 13.

The observed practical minimum ranged from eight to 17 in one casino and from 20 up in another. The theoretical minimum, when no cards are burned, is six. Thus the results for 13 unplayed cards seem to conclusively demonstrate that no *practical* winning strategy is possible for the Nevada game, even with a computing machine playing a perfect game.

To see why, consider the accompanying Table 3–2 (pp. 34-35), based on Table 2 of Walden's thesis.

From this Table we see the effect of removing one of any card from the eight-deck baccarat pack. Proceeding in the way we developed the theory of blackjack, we get relative point values which are listed in the next to the last column. The last column gives a simpler approximate point count system.

We would now like to know how powerful a point count system in baccarat is compared with point count systems in blackjack. To do this we compute the root mean square (RMS) value of the column called "Change in Advantage of Banker Bet."

We do this by squaring each of those numbers, counting the square for zero-value cards four times because there are four times as many. Then we add these squares, divide by 13 and take the square root. The resulting root mean square or RMS value is .0064%. That measures how fast the deck shifts from its base starting value for a full pack.

Taking one card of a given rank (we think of there being 13 ranks) changes the fraction of any of these 13 ranks by an amount 32/416-31/415 which equals .00222. If we divide the RMS value by this value we get .0288 as a measure of how rapidly this advantage of the two bets shifts from the starting value as the composition of the deck changes.

Now we are going to compare this with the situation in blackjack. Table 3-3, for one-deck blackjack, can be treated in the same way to see how fast the advantage changes in blackjack.

The Table is from Peter Griffin's book, *Theory of Blackjack Revised*, page 44. We get RMS value of 0.467%. The corresponding change in the fraction of a single rank when one card is drawn

Table 3-3
One-Deck Blackjack

Subtract One Card of Value	Change in Advantage of Player
A	−0.61%
2	0.38%
3	0.44%
4	0.55%
5	0.69%
6	0.46%
7	0.28%
8	−.00%
9	−.18%
10	.51%

is 4/52-3/51 or .0181. The ratio of RMS value to this value is .258%. If we divide this by the corresponding result in baccarat we get 8.97, which tells us that as the true count in blackjack varies the change in player advantage or disadvantage shifts nine times as fast in blackjack as it does in baccarat.

Note that dividing the RMS value by the "change in fraction" of a single pack adjusts the one-deck blackjack figures and the eight-deck baccarat figures so they are comparable. If we had used, for example, an eight-deck blackjack table instead, we still would have had a final ratio of about nine times.

This allows us to translate how well a point count in baccarat works compared with one in blackjack. In baccarat we start out with more than a 1% disadvantage and with eight decks. Imagine a blackjack game with an eight-deck pack and a 1% disadvantage. Now imagine play continues through the blackjack deck. The blackjack deck advantage from a − 1% starting level might, on very rare occasions, shift 9% to a + 8% advantage.

As often as this happens in blackjack would be the approximate frequency with which we would get 1/9th as much shift in baccarat; meaning from a − 1% advantage to a 0% advantage or break even for the banker bet. Since there are two bets, banker and player, the player bet would also be break even or better about as often.

The conclusion is that you might expect to break even or better in eight-deck baccarat about twice as often as you would expect to have an 8% edge in eight-deck blackjack. How often would you have a 1% advantage in eight-deck baccarat? About twice as often as you would get a 17% edge in blackjack. The obvious conclusion is that advantages in baccarat are very small, they are very rare and the few that occur are nearly always in the last five to 20 cards in the pack.

The Tie Bet

In addition to wagers on the Player or Banker hands, the casinos offer a bet on "ties." In the event the Banker and Player hands have the same total, this bet gains nine times the amount bet. Otherwise the bet is lost. The probability of a tie is 9.5156%, hence the expectation of the bet is −4.884%.

It is clear, however, that the probability and thus the expectation of a tie depends on the subset of unplayed cards. For instance, in the extreme and improbable event that the residual deck

consists solely of ten-value cards, the probability of a tie is equal to one and the expectation is nine. Thus card counting strategies are potentially advantageous.

Using computer simulation, random subsets of different sizes were selected from a complete 416-card (eight deck) pack. The results were disappointing from a money-making perspective—the advantages which occur with complete knowledge of the used cards are limited to the extreme end of the pack and are generally not large. Practical card counting strategies are at best marginal, and at best precarious, for they are easily eliminated by shuffling the deck with 26 cards remaining.