Hedging and Leveraging: Principal Portfolios of the Capital Asset Pricing Model

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The principal portfolios of the standard Capital Asset Pricing Model (CAPM) are analyzed and found to have remarkable hedging and leveraging properties. Principal portfolios implement a recasting of any *correlated* asset set of N risky securities into an equivalent but *uncorrelated* set when short sales are allowed. While a determination of principal portfolios in general requires a detailed knowledge of the covariance matrix for the asset set, the rather simple structure of CAPM permits an accurate solution for any reasonably large asset set that reveals interesting universal properties. Thus for an asset set of size N, we find a *market-aligned* portfolio, corresponding to the *market* portfolio of CAPM, as well $N-1$ *market-orthogonal* portfolios which are market neutral and strongly leveraged. These results provide new insight into the return-volatility structure of CAPM, and demonstrate the effect of unbridled leveraging on volatility.

1. INTRODUCTION

Modern investment theory dates back to the mean-variance analysis of Markowitz (1952, 1959), which is expected to hold if asset prices are normally distributed or the investor preferences are quadratic. Undoubtedly, the most consequential fruit of Markowitz' seminal work was the introduction of the capital asset pricing model (CAPM) by Sharpe (1964), Lintner (1965), and Mossin (1966). The key ideas of this model are that investors are mean-variance optimizers facing a frictionless market with full agreement on the distribution of security returns and unrestricted access to borrowing and lending at the riskless rate. As an asset pricing model, CAPM is an equilibrium model valid for a given investment horizon, which is taken to be the same for all investors. Indeed investors are solely distinguished by their level of risk aversion.

Principal portfolio analysis, on the other hand, simplifies asset allocation by recasting the asset set into uncorrelated portfolios when short sales are allowed (Partovi and Caputo 2004). Stated otherwise, the original problem of stock selection from a set of *correlated assets* is transformed into the much simpler problem of choosing from a set of *uncorrelated portfolios*. The details of this transformation are given in Partovi and Caputo (2004), where the results are summarized as follows: *Every investment environment* $\{s_i, r_i, \sigma_{ij}\}_{i,j=1}^{N}$ *which allows short sales can be recast as a principal portfolio environment* $\{S_\mu, R_\mu, \mathsf{V}_{\mu\nu}\}_{\mu,\nu=1}^{N}$ *where the principal covariance matrix* V *is diagonal. The weighted mean of the principal variances equals the mean variance of the original environment. In general, a typical principal portfolio is hedged and leveraged.* Here s_i (S_μ), r_i (R_μ), and σ_{ij} ($\mathsf{V}_{\mu\nu}$) represent the assets, the expected returns, and the covariance matrix of the original (recast) set, while N is the size of the asset set. It was further shown in Partovi and Caputo (2004) that the efficient frontier in the presence of a riskless asset has a simple allocation rule which requires that each principal portfolio be included in inverse proportion to its variance. Practical applications of principal portfolios have already been considered by several authors, for example, Poddig and Unger (2012) and Kind (2013).

In this paper we present a perturbative calculation of the principal portfolios of the single-index CAPM in the large N limit. The results of this calculation are in general expected to entail a relative error of the order of $1/N^2$. However, since any application of the single-index CAPM is most likely to involve a large asset set, the stated error is normally quite small and in any case majorized by modelling errors. Thus the results to be reported here are accurate implications of the underlying model.

The principal portfolio analysis of the single-index model and an exactly solvable version of it presented in §3 highlight the volatility structure of principal portfolios in a practical and familiar context. A remarkable result of the analysis is the bifurcation of the set of principal portfolios into a *market-aligned* portfolio, which is unleveraged and behaves rather like a total-market index fund, and $N-1$ *market-orthogonal* portfolios, which are hedged and leveraged,[1] and nearly free of market driven fluctuations. This equivalency between the original asset set and two classes of principal portfolios is reminiscent of, but fundamentally different from, Merton's (1972) two mutual fund theorems. The market-orthogonal portfolios, on the other hand, provide a vivid demonstration of the effect of leveraging on the volatility level of a portfolio.

2. PRINCIPAL PORTFOLIOS OF THE SINGLE-INDEX MODEL

Here we shall analyze the standard single-index model as well as an exactly solvable special case of it with respect to their principal portfolio structure. Remarkably, our analysis will uncover interesting and hitherto unnoticed properties of well-diversified and arbitrarily leveraged portfolios within the single-index model.

Consider a set of N assets $\{s_i\}$, $1 \le i \le N$, whose rates of return are normally distributed random variables given by

$$\rho_i \overset{\text{def}}{=} \alpha_i + \beta_i \rho_{mkt}, \tag{1}$$

where α_i and ρ_{mkt} are uncorrelated, normally distributed random variables with expected values and variances equal to $\bar{\alpha}_i$, $\bar{\rho}_{mkt}$ and $\overline{\alpha_i^2}$, $\overline{\rho^2}_{mkt}$, respectively. The quantity β_i associated with asset s_i is a constant which measures the degree to which s_i is coupled to the overall market variations. Thus the attributes of a given asset are assumed to consist of a *market-driven* (or *systematic*) part described by $(\beta_i \rho_{mkt}, \beta_i^2 \overline{\rho^2}_{mkt})$ and a *residual* (or *specific*) part described by $(\alpha_i, \overline{\alpha_i^2})$, with the two parts being uncorrelated.

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[1] We use the term "leveraged" here to imply that the portfolio contains borrowed assets, e.g., short-sold positions.

The expected value of Eq. (1) is given by

$$\bar{\rho}_i \stackrel{\text{def}}{=} r_i = \bar{\alpha}_i + \beta_i \bar{\rho}_{mkt}. \tag{2}$$

The covariance matrix which results from Eq. (1) is similarly a superposition of the specific and market-driven contributions, as would be expected of the sum of two uncorrelated variables. It can be written as

$$\sigma_{ij} = \overline{\alpha_i^2}\delta_{ij} + \beta_i\beta_j\overline{\rho^2}_{mkt}. \tag{3}$$

Note that σ is a *definite* matrix, since we have excluded riskless assets from the asset set for the time being.

We shall assume here that the number of assets N is appropriately large, as is in fact implicit in the formulation of all index models, so that the condition $\overline{\alpha_i^2}/Nb^2\overline{\rho^2}_{mkt} \ll 1$ is satisfied; here $b \stackrel{\text{def}}{=} (\sum_{i=1}^N \beta_i^2/N)^{\frac{1}{2}}$ is the square root of the average value of β_i^2, typically of the order of unity. These assumptions are not essential to our discussion, but they do simplify the presentation and more importantly, they are usually well satisfied for appropriately large values of N and guarantee that our purturbative results below are accurate for practical applications.

Under the above assumptions it is appropriate to rescale the covariance matrix as in $\sigma_{ij} = Nb^2\overline{\rho^2}_{mkt}\tilde{\sigma}_{ij}$, where

$$\tilde{\sigma}_{ij} \stackrel{\text{def}}{=} \gamma_i^2\delta_{ij} + \hat{\beta}_i\hat{\beta}_j \tag{4}$$

is a dimensionless matrix. Here $\hat{\beta}_i \stackrel{\text{def}}{=} \beta_i/(\sum_{i=1}^N \beta_i^2)^{\frac{1}{2}}$, so that $\hat{\boldsymbol{\beta}} = (\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_N)$ is a unit vector, and $\gamma_i^2 \stackrel{\text{def}}{=} \overline{\alpha_i^2}/Nb^2\overline{\rho^2}_{mkt} \ll 1$ as concluded above.

The above representation of the covariance matrix for the single-index model is quite suitable for revealing the structure of its eigenvalues and eigenvectors, these being closely related to the rescaled principal variances and the principal portfolios we are seeking. The structure in question is actually discernible on the basis of simple, qualitative considerations of the spectrum of $\tilde{\sigma}$. To see this structure, let us first note that the sum of the eigenvalues of $\tilde{\sigma}$, which is given by $\mathrm{tr}(\tilde{\sigma}) \stackrel{\text{def}}{=} \sum_{i=1}^N \tilde{\sigma}_{ii}$, equals $1+\sum_{i=1}^N \gamma_i^2$. We will show below that the largest eigenvalue of $\tilde{\sigma}$ is approximately equal to unity, so that the remaining $N-1$ eigenvalues have an average value approximately equal to the average of $\{\gamma_i^2\}$, which was shown above to be much smaller than unity as a consequence of the large N assumption. Thus, barring a strongly skewed distribution of the latter, which is all but ruled out for any of the customary asset classes, we find that the spectrum of $\tilde{\sigma}$ consists of a "major" eigenvalue close to unity, and $N-1$ "minor" eigenvalues each much smaller than unity. Stated in terms of the spectrum of V, this implies that the principal portfolios separate into two classes of quite different properties, namely (i) a single *market-aligned* portfolio with a variance of magnitude approximately equal to $Nb^2\overline{\rho^2}_{mkt}/{W_N}^2$, and (ii) $N-1$ *market-orthogonal* portfolios whose variances have a weighted average approximately equal to the average of the residual variance of the original asset set. As one might suspect, these two categories are characterized by sharply different values of portfolio beta,[2] the former with a value typical of the asset set (i.e., of the order of unity) and the remaining $N-1$ portfolios with much smaller (possibly vanishing; cf. §3) values.

To see the quantitative details of the foregoing qualitative analysis, we now turn to a perturbative treatment of the spectrum of $\tilde{\sigma}$. The eigenvalue equation for $\tilde{\sigma}$ reads

$$(\tilde{\sigma}\mathbf{e}^\mu)_i = \gamma_i^2 e_i^\mu + \hat{\boldsymbol{\beta}}\cdot\mathbf{e}^\mu\hat{\beta}_i = \tilde{v}_\mu^2 e_i^\mu, \tag{5}$$

where $\boldsymbol{e}^\mu$ is the μth eigenvector, e_i^μ is the ith component of that eigenvector, and $\tilde{v}_\mu^2$ is the corresponding eigenvalue, all quantities as defined earlier. Because of its simple structure, Eq. (5) can be implicitly solved for the components of the eigenvectors to yield

$$e_i^\mu = [\hat{\boldsymbol{\beta}}\cdot\mathbf{e}^\mu/(\tilde{v}_\mu^2 - \gamma_i^2)]\hat{\beta}_i. \tag{6}$$

Upon multiplying this equation by $\hat{\beta}_i$ and summing over i, we find the characteristic equation for the eigenvalues. It reads

$$1 = \sum\nolimits_{i=1}^N [\hat{\beta}_i^2/(\tilde{v}_\mu^2 - \gamma_i^2)]. \tag{7}$$

This equation can be rearranged as an Nth-order polynomial equation in the variable $\tilde{v}_\mu^2$, the μth eigenvalue of σ divided by $Nb^2\overline{\rho^2}_{mkt}$, and is guaranteed to have N real, positive roots (with multiple roots counted according to

[2] Here the portfolio beta is defined to be the weighted mean of beta in the single-index model literature

their multiplicity). Once these roots are determined, they can be used in Eq. (6) to find the eigenvectors in the usual manner.

As mentioned earlier, the structure of $\tilde{\sigma}$ allows an approximate determination of its largest eigenvalue when N is suitably large, say a hundred or more. This is of course a significant advantage in any numerical solution of the equations described in the preceding paragraph. As one can see from Eq. (4), the matrix in question, $\tilde{\sigma}$, is the sum of two parts, one is diagonal with elements γ_i^2 which are much smaller than unity, and the other a rank-1 matrix with eigenvalue equal to unity. This implies that the eigenvector of the latter matrix is an approximate eigenvector of $\tilde{\sigma}$ with eigenvalue approximately equal to unity. This is the eigenvalue we designated as *major* in our qualitative discussion. Let this be the Nth eigenvalue, so that $\tilde{v}_N^2 \stackrel{\text{def}}{=} 1+\epsilon_N$, with $|\epsilon_N| \ll 1$. Substituting this expression for $\tilde{v}_N^2$ in Eq. (7), and treating the resulting equation to first order in γ_i^2,[3] we find

$$\tilde{v}_N^2 \simeq 1+\sum\nolimits_{i=1}^{N}\gamma_i^2\hat{\beta}_i^2, \tag{8}$$

which identifies ϵ_N as equal to $\sum_{i=1}^{N}\gamma_i^2\hat{\beta}_i^2$ to first order, thus verifying the condition $|\epsilon_N| \ll 1$. The corresponding eigenvector can now be found from Eq. (6); to first order, it is given by

$$e_i^N \simeq (1+\gamma_i^2-\sum\nolimits_{j=1}^{N}\gamma_j^2\hat{\beta}_j^2)\hat{\beta}_i, \tag{9}$$

where the conditions of unit length and non-negative relative weight stipulated earlier have already been imposed within the stated order of approximation.

Equation (9) specifies the (relative) composition of the market-aligned portfolio. The relative weight W_N of this portfolio, on the other hand, is expected to be of the order of $N^{\frac{1}{2}}$, since this portfolio consists entirely of purchased assets (recall our estimate of the relative weights earlier in §2). Indeed one can see from Eq. (9) that $W_N \simeq \sum_{i=1}^{N}\hat{\beta}_i$ in the leading order of approximation,[4] which confirms the above-stated estimate (recall that the average of the $\hat{\beta}_i^2$ equals N^{-1}). Equations (8) and (9) provide approximate expressions for the major eigenvalue and eigenvector of the covariance matrix of the single-index model.

Rescaling Eqs. (8) and (9) back to original variables, we find, for the variance and the composition of the market-aligned principal portfolio, the expressions

$$V_N{}^2 \simeq [1+3\sum\nolimits_{i=1}^{N}\gamma_i^2\hat{\beta}_i^2-(\sum\nolimits_{i=1}^{N}\hat{\beta}_i)^{-1}\sum\nolimits_{i=1}^{N}\gamma_i^2\hat{\beta}_i](\boldsymbol{\beta}\cdot\boldsymbol{\beta})\bar{\rho^2}_{mkt}/(\sum\nolimits_{i=1}^{N}\hat{\beta}_i)^2, \tag{10}$$

$$e_i^N/W_N \simeq [1+\gamma_i^2-(\sum\nolimits_{i=1}^{N}\hat{\beta}_i)^{-1}\sum\nolimits_{i=1}^{N}\gamma_i^2\hat{\beta}_i]\hat{\beta}_i/(\sum\nolimits_{j=1}^{N}\hat{\beta}_j), \tag{11}$$

where we have left the small correction terms in dimensionless form. It is clear from Eq. (11) that the market-aligned portfolio is basically composed by investing in each asset in proportion to how strongly it is correlated with the overall market fluctuations, i.e., in proportion to the value of its beta; cf. Eq. (1). Consequently, it is expected to be strongly suseptible to market-driven fluctuations. Indeed as one can see from Eq. (10), the variance of this principal portfolio in the leading order is given by $(Nb^2/W_N{}^2)\bar{\rho^2}_{mkt}$, which is of the same order of magnitude as $\bar{\rho^2}_{mkt}$ (recall that b is of the same approximate magnitude as a typical β and that W_N is of the order of $N^{\frac{1}{2}}$). The market-aligned portfolio is therefore seen to be that principal portfolio which approximately reflects the volatility profile of the market as a whole. Moreover, since it entirely composed of purchased assets, it is neither hedged nor leveraged.

By contrast, the remaining $N-1$ market-orthogonal principal portfolios are in general hedged and leveraged, and they are quite immune to overall market fluctuations[5]. In fact, since $\sum_{\mu=1}^{N}v_\mu^2=\mathrm{tr}(\sigma)=\boldsymbol{\beta}\cdot\boldsymbol{\beta}\bar{\rho^2}_{mkt}(1+\sum_{i=1}^{N}\gamma_i^2)$, and $v_N^2 \simeq \boldsymbol{\beta}\cdot\boldsymbol{\beta}\bar{\rho^2}_{mkt}+\sum_{i=1}^{N}\hat{\beta}_i^2\bar{\alpha_i^2}$, we find for the the average value of the $N-1$ minor eigenvalues

$$(N-1)^{-1}\sum\nolimits_{\mu=1}^{N-1}v_\mu^2=(N-1)^{-1}\sum\nolimits_{\mu=1}^{N-1}W_\mu{}^2V_\mu{}^2 \simeq (N-1)^{-1}\sum\nolimits_{i=1}^{N}(1-\hat{\beta}_i^2)\bar{\alpha_i^2}. \tag{12}$$

Thus the weighted average of principal variances for market-orthogonal portfolios is approximately equal to (and in fact less than) the average of the residual variances of the original asset set. Therefore, these $N-1$ market-orthogonal

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[3] This is the approximation in which any contribution to $\tilde{v}_N^2$ whose ratio to γ_i^2 vanishes in the $N\to\infty$ limit will be neglected.
[4] This is the approximation in which any contribution to W_N whose ratio to $\hat{\beta}_i$ vanishes in the $N\to\infty$ limit will be neglected.
[5] These market-orthogonal portfolios essentially eliminate what is referred to as "market risk" in the single-index model jargon

principal portfolios are free not only of mutual correlations with other portfolios but in general also of the volatility induced by overall market fluctuations. This feat is possible in part because of the very special structure of the single-index model which makes it possible to isolate essentially all of the systematic market volatility in one portfolio, leaving the remaining $N-1$ portfolios almost totally immune to systematic market fluctuations.

There is an important caveat with respect to the foregoing statement. Recall that there is an inverse relationship between V_μ, defined as the positive square root of V_μ^2, and W_μ, so that for highly leveraged portfolios which are characterized by the condition $W_\mu \ll 1$, the above argument would imply a principal variance far exceeding the original ones. Of course the condition $W_\mu \ll 1$ that implies such large variances also implies large expected returns, so that a more sensible comparative measure under such conditions is $\check{V}_\mu \stackrel{\text{def}}{=} V_\mu/R_\mu = v_\mu/\sum_{i=1}^N e_i^\mu r_i$, which may be called *return-adjusted volatility* of the principal portfolio. As expected, the relative weight W_μ is no longer present in this adjusted version of the volatility.

The return-adjusted volatility for the market-aligned portfolio, on the other hand, can be calculated from Eqs. (1), (10), and (11). It is given by

$$\check{V}_N \simeq \{1 - [(\bar{\rho^2}_{mkt})^{\frac{1}{2}}/\bar{\rho}_{mkt}]\sum_{i=1}^N \gamma_i \hat{\beta}_i\}(\bar{\rho^2}_{mkt})^{\frac{1}{2}}/\bar{\rho}_{mkt}, \tag{13}$$

which is approximately equal to $(\bar{\rho^2}_{mkt})^{\frac{1}{2}}/\bar{\rho}_{mkt}$. This ratio is of course precisely what one would expect for the approximate value of the return-adjusted volatility of a portfolio which is aligned with the overall market price movements.

It is appropriate at this point to summarize the properties of the principal portfolios for the single-index model.

Proposition 1. *The principal portfolios of the single-index model consist of a market-aligned portfolio, which is unleveraged and has a return-adjusted volatility* $\simeq (\bar{\rho^2}_{mkt})^{\frac{1}{2}}/\bar{\rho}_{mkt}$ *characteristic of market-driven price movements, and* $N-1$ *market-orthogonal portfolios which are hedged and leveraged, and nearly free of systematic market fluctuations. Equations (10)-(13) provide approximate expressions valid to first order in* $1/N$ *for the properties of these portfolios.*

3. SINGLE-INDEX MODEL WITH CONSTANT RESIDUAL VARIANCE

To provide an explicit illustration of the principal portfolio structure within the single-index model described in the preceding section, we now turn to an exactly solvable, albeit oversimplified, version of that model. This model is defined by the assumption that the residual variance of the ith asset in the original set, $\bar{\alpha_i^2}$, is equal to $\bar{\alpha^2}$ for all assets. Observe that this assumption does not affect the expected rate of return for the ith asset, which is given by $r_i = \bar{\alpha}_i + \beta_i \bar{\rho}_{mkt}$ as before. This simplification will allow us to derive an exact solution for the model and illustrate the concepts and methods of the previous section in more explicit terms. The price for this simplification is of course the unrealistic assumption of constant residual variance which defines the model.

The covariance matrix with the above simplification appears as

$$\sigma_{ij}^{crv} = \bar{\alpha^2}\delta_{ij} + \beta_i\beta_j\bar{\rho^2}_{mkt}, \tag{14}$$

whose rescaled version is

$$\tilde{\sigma}_{ij}^{crv} = \gamma^2\delta_{ij} + \hat{\beta}_i\hat{\beta}_j \tag{15}$$

These equations are of course specialized versions of Eqs. (3) and (4).

Referring to the results of the previous section, one can readily see that the spectrum of $\tilde{\sigma}^{crv}$ consists of a major eigenvalue (exactly) equal to $1+\gamma^2$ [cf. Eq. (8)], and $N-1$ minor eigenvalues, all equal to γ^2. Recall that these eigenvalues respectively correspond to the market-aligned and market-orthogonal portfolios introduced in §4.1. Not surprisingly, the spectrum of $\tilde{\sigma}_{ij}^{crv}$ is found to be highly degenerate. The eigenvector $\mathbf{e}^{crvN}$ corresponding to the major eigenvalue is (exactly) equal to $\hat{\beta}_i$ [cf. Eq. (9)], while the remaining $N-1$ minor eigenvectors are not uniquely determined[6] and may be arbitrarily chosen to be any orthonormal set of $N-1$ vectors orthogonal to the major eigenvector $\hat{\beta}_i$. The expected return and volatility features of the $N-1$ market-orthogonal portfolios defined by this arbitrary choice, on the other hand, do depend on that choice, as the following analysis will show.

Since our main objective is the determination of the efficient frontier, we shall choose the remaining $N-1$ eigenvectors with respect to their volatility level, which, it may be recalled from §2, is given by ${V_\mu}^2 = v_\mu^2/{W_\mu}^2$. For the

[6] This is of course the exceptional case of spectrum degeneracy mentioned in §2.

present case, minimizing V_μ amounts to maximizing W_μ. Therefore, we will look for a unit vector $\mathbf{e}$ that is orthogonal to $\hat{\boldsymbol{\beta}}$ as stipulated above *and* maximizes $\sum_{i=1}^{N} e_i$. In terms of rescaled quantities, this problem appears as

$$\max{}_{\mathbf{e}}\, \mathbf{e}\cdot\hat{\mathbf{u}} \;\; s.t.\, \mathbf{e}\cdot\mathbf{e}=1,\;\; \mathbf{e}\cdot\hat{\boldsymbol{\beta}}=0, \tag{16}$$

where $\hat{\mathbf{u}}_i \stackrel{\text{def}}{=} N^{-\frac{1}{2}}(1,1,\ldots,1)$ is an N-dimensional unit vector all of whose components are equal. The solution to Eq. (16) may be found by standard methods provided that $\hat{\mathbf{u}}$ and $\hat{\boldsymbol{\beta}}$ are not parallel, a condition whose violation is very improbable and will henceforth be assumed to hold. On the other hand, it is clear from geometric considerations that the solution must be that linear combination of $\hat{\mathbf{u}}$ and $\hat{\boldsymbol{\beta}}$ which is orthogonal to $\hat{\boldsymbol{\beta}}$. Designating the solution vector as $\mathbf{e}^{crv1}$, we find

$$\mathbf{e}_i^{crv1} = [\hat{u}_i - \cos(\theta)\hat{\beta}_i]/\sin(\theta), \tag{17}$$

where θ is the angle formed by the unit vectors $\hat{\mathbf{u}}$ and $\hat{\boldsymbol{\beta}}$, constrained by the condition $0<\theta\le\pi/2$ under our assumptions. Indeed a little algebra shows that

$$\tan(\theta) = \delta\beta/\bar{\beta}, \tag{18}$$

where $\bar{\beta}$ and $\delta\beta$ respectively denote the mean and the standard deviation of the β's, i.e., $N^{-1}\sum_{i=1}^{N}\beta_i$ and $\left[N^{-1}\sum_{i=1}^{N}(\beta_i-\bar{\beta})^2\right]^{\frac{1}{2}}$. Equation (18) clearly shows that the angle θ represents the degree of scatter among the betas, vanishing when all betas are equal and increasing as they are made more unequal. Note that the condition $\theta>0$ stipulated above excludes the (improbable) case of uniform betas. Note also that the condition $\mathbf{e}\cdot\hat{\boldsymbol{\beta}}=0$ in Eq. (16) is equivalent to the vanishing of the portfolio beta for $\mathbf{e}_i^{crv1}$, in contrast to the same quantity for the market-aligned portfolio which is found to be $\bar{\beta}/\cos^2(\theta)$.

Thus far we have determined two eigenvectors, the market-aligned $\mathbf{e}^{crvN}$, and the minimum-volatility, market-orthogonal eigenvector $\mathbf{e}^{crv1}$. The remaining $N-2$ market-orthogonal eigenvectors will of course have to be orthogonal to these, which immediately implies that they will all be orthogonal to $\hat{\mathbf{u}}$. But orthogonality to $\hat{\mathbf{u}}$ implies a vanishing weight, thus implying that these portfolios require zero initial investment. Stated differently, these $N-2$ portfolios are *critically* leveraged, with the short-sold assets precisely balancing the purchased ones in each portfolio. Under these circumstances, any volatility in portfolio return would imply an infinite variance for that principal portfolio because of the vanishing initial investment. Note that the return-adjusted volatility for these portfolios, on the other hand, need not (and in typical cases will not) diverge at all.

As stated earlier, the efficient frontier in the presence of a riskless asset has a simple allocation rule which requires that each principal portfolio be included in inverse proportion to its variance. For the current case, this rule clearly excludes the $N-2$ portfolios described above from the efficient frontier, leaving the first two principal portfolios and the riskless asset as the only constituents. Thus for the special case of constant residual variance, a knowledge of the two distinguished principal portfolios determined above is all that is needed to specify the efficient frontier when a riskless asset is present. For this reason, we will not continue with the explicit construction of the remaining $N-2$ eigenvectors.

At this point we can determine the expected value and the variance of the two principal portfolios determined above according to the definitions and formulae given in §2. Straightforward algebra leads to

$$R_N^{crv} = \frac{\sum_{i=1}^{N}\hat{\beta}_i(\bar{\alpha}_i+\beta_i\bar{\rho}_{mkt})}{N^{\frac{1}{2}}\cos(\theta)},\;\; (V_N^{crv})^2 = \frac{\bar{\alpha^2}+\boldsymbol{\beta}\cdot\boldsymbol{\beta}\bar{\rho^2}_{mkt}}{N\cos^2(\theta)}, \tag{19}$$

for the market-aligned portfolio, and

$$R_1^{crv} = \frac{r^{av}}{\sin^2(\theta)} - \cot^2(\theta)R_N^{crv},\;\; (V_1^{crv})^2 = \frac{\bar{\alpha^2}}{N\sin^2(\theta)}, \tag{20}$$

for the market-orthogonal, minimum volatility principal portfolio. In order to facilitate comparison with the perturbative results of §2 for the general single-index model, we also record here the return-adjusted volatilities of these portfolios;

$$\check{V}_N^{crv} = \frac{(\bar{\alpha^2}+\boldsymbol{\beta}\cdot\boldsymbol{\beta}\bar{\rho^2}_{mkt})^{\frac{1}{2}}}{\sum_{i=1}^{N}\hat{\beta}_i(\bar{\alpha}_i+\beta_i\bar{\rho}_{mkt})}, \tag{21}$$

$$\check{V}_1^{crv} = \frac{[\bar{\alpha^2}\sin^2(\theta)]^{\frac{1}{2}}}{N^{-\frac{1}{2}}[r^{av} - \cos^2(\theta) R_N^{crv}]}. \tag{22}$$

The results just derived demonstrate the powerful volatility reduction effect of diversification coupled with short sales for the market-orthogonal portfolio $\mathbf{e}^{crv1}$. To see this, let us assume a typical value for $\tan(\theta)$ of the order of unity [for reasonably large N; cf. Eq. (18)]. We then find from the above results

$$\lim_{N\to\infty} V_N^{crv} = [\frac{\boldsymbol{\beta}\cdot\boldsymbol{\beta}}{N\cos^2(\theta)}\bar{\rho^2}_{mkt}]^{\frac{1}{2}},\ \lim_{N\to\infty} V_1^{crv} = 0. \tag{23}$$

Note that the quantity $\boldsymbol{\beta}\cdot\boldsymbol{\beta}$ in general grows in proportion to N, and therefore that $\boldsymbol{\beta}\cdot\boldsymbol{\beta}/N\cos^2(\theta)$ is typically of the order of unity for large N. Thus the variance of the market-aligned portfolio will be of the order of $\bar{\rho^2}_{mkt}$ for large N, as would be expected. The variance of the market-orthogonal portfolio, on the other hand, vanishes altogether in proportion to N^{-1} in the same limit of large N. These conclusions echo our results in §2, Eq. (13) et seq.

Note that the vanishing of the market risk for the market-orthogonal portfolio, which is in addition to the vanishing of the "diversifiable" (or specific) risk expected for large N (Elton and Gruber 1991), is a specific result of leveraging coupled with hedging (or diversification). Similarly, the infinite volatility and expected return levels of the $N-2$ remaining portfolios of this model underscore the dramatic levels of volatility as well as return that can be expected of highly leveraged portfolios.

We are now in a position to determine the composition of the efficient frontier for the constant residual variance case. As stated above, we find from the allocation rule of the efficient frontier that ${X^{crv}}_\mu = 0$ for $2 \le \mu \le N-1$, since the corresponding inverse variances ${Z^{crv}}_\mu$ all vanish. The three components of the efficient frontier are the riskless asset together with $\mathbf{e}^{crv1}$ and $\mathbf{e}^{crvN}$. Furthermore, the latter portfolio will be strongly disfavored relative to the former for large N since its variance grows in proportion to N relative to that of the former; cf. Eq. (23) et seq. Indeed for reasonably large N, the efficient frontier is essentially a combination of $\mathbf{e}^{crv1}$ and the riskless asset;

$$X_0^{crv} \rightarrow \frac{R_1^{crv} - \mathcal{R}}{R_1^{crv} - R_0},\ X_1^{crv} \rightarrow \frac{\mathcal{R} - R_0}{R_1^{crv} - R_0},\ \text{ as } N \to \infty, \tag{24}$$

while

$$X_N^{crv} \rightarrow 0,\ V_{eff} = |\frac{\mathcal{R} - R_0}{R_1^{crv} - R_0}|[\frac{\bar{\alpha^2}}{N\sin^2(\theta)}]^{\frac{1}{2}} \rightarrow 0 \text{ as } N \to \infty. \tag{25}$$

This last property, i.e., the vanishing of the efficient portfolio volatility (i.e., the market as well as the specific risk) in proportion to $N^{-\frac{1}{2}}$ in the limit as $N \to \infty$, also holds for the general single-index model, as can be discerned from the results of §2. As discussed earlier, this total vanishing of the portfolio volatility is a specific consequence of leveraging.

We close this section by summarizing the results established above.

Proposition 2. *The principal portfolios of the constant variance single-index model consist of a market-aligned portfolio* $\mathbf{e}^{crvN}$*, a minimum-volatility, market-orthogonal portfolio* $\mathbf{e}^{crv1}$*, and* $N-2$ *critically leveraged market-orthogonal portfolios with infinite volatility and expected return, as given in Eqs. (17)-(23). Furthermore, as* $N \to \infty$*, the efficient portfolio reduces to a combination of* $\mathbf{e}^{crv1}$ *and the riskless asset with a vanishing total volatility, as given in Eqs. (24)-(25).*

4. CONCLUDING REMARKS

The main objective of this work, namely recasting the efficient portfolio problem in terms of principal portfolios whereby the selection is made from an uncorrelated set of portfolios instead of the original asset set has been implemented in detail. As we have emphasized throughout, principal portfolios are the natural instruments for analyzing the efficient frontier when short sales are allowed. More generally, they are the natural instruments for any stock selection process based on the mean-variance formulation. In effect, the analysis is transformed from the original, correlated environment of individual assets to one of uncorrelated portfolios, thus simplifying both the conceptual framework as well as the practical procedures involved. This is of course another example of the golden rule in applied analysis which teaches us that when the basic object in the problem involves a quadratic form, it is often advantageous to treat it in the principal axes basis.

In order to illustrate the concepts and methods of this paper, we have analyzed the single-index model as well as the constant residual variance version of it in considerable detail. Indeed our perturbative treatment of the general single-index model for large asset sets has revealed interesting new spectral features for that model. In particular, the bifurcation into market-aligned and market-orthogonal portfolios found in §2 is an important observation on the volatility structure of the model. This is particularly so in view of the fact that for sufficiently large asset sets the market-aligned portfolio is all but excluded from the efficient frontier thereby eliminating the component of volatility commonly referred to as market risk. The constant residual variance version of the model, while admittedly oversimplified, brings out the above-mentioned bifurcation as well as the elimination of the market risk in a clear and explicit manner.

The virtual elimination of efficient portfolio volatility for reasonably large asset sets as well as the occurrence of large volatility and expected return principal portfolios encountered in our analysis demonstrate the combined effects of leveraging and diversification. As an example, consider the market-aligned portfolio in the single-index model. This portfolio does not involve any short positions and is unleveraged. Consequently, its volatility is characteristic of the market volatility and does not vanish for large asset sets. The market-orthogonal portfolios, on the other hand, are leveraged to varying degrees, a feature that (together with hedging) largely immunizes them against the market volatility. These features are amplified and explicitly demonstrated by the two distinguished principal portfolios of the constant residual variance model. The remaining portfolios of this model, it may be recalled, are critically leveraged and as such have infinite volatility and expected return.

We conclude by observing that the mean-variance description of risky asset prices whereby short-term price variations are taken to be random fluctuations has been a remarkably fruitful idea for describing the dynamics of financial markets, its well known limitations notwithstanding. We have attempted in this work to extend the utility of that idea by providing a new analytical tool for its implementation.

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