News Trading and Speed^{*}

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Abstract

Informed trading can take two forms: (i) trading on more accurate information or (ii) trading on public information faster than other investors. The latter is increasingly important due to technological advances. To disentangle the effects of accuracy and speed, we derive the optimal dynamic trading strategy of an informed investor when he reacts to news (i) at the same speed or (ii) faster than other market participants, holding information precision constant. With a speed advantage, the informed investor's order flow is much more volatile, accounts for a much bigger fraction of trading volume, and forecasts very short run price changes. We use the model to analyze the effects of high frequency traders on news (HFTNs) on liquidity, volatility, price discovery and provide empirical predictions about the determinants of their activity.

KEYWORDS: Informed trading, news, high frequency trading, liquidity, volatility, price discovery.

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1 Introduction

The effect of news arrival on trades and prices in securities markets is of central interest. For instance, informational efficiency is often measured by the speed at which prices incorporate public information and many researchers have studied trading volume and prices around news (e.g., Patell and Wolfson (1984), Kim and Verrecchia (1991, 1994), Busse and Green (2001), Vega (2006), or Tetlock (2010)). A new breed of market participants, "high frequency traders on news" (HFTNs), now use the power of computers to collect, process and exploit news faster than other market participants (see "Computers that trade on the news", the New York Times, May 2012).¹ Hence, the impact of news in today's securities markets depends on the behavior of these traders. Can we rely on traditional models of informed trading to understand this behavior and its effects? Is trading faster on public information the same thing as trading on more accurate private information?

To address these questions, we consider a model in which an informed investor continuously receives news about the payoff of a risky security. He has both a greater information processing capacity *and* a higher speed of reaction to news than market makers. The information processing advantage enables the informed investor to form a more precise forecast of the fundamental value of the asset while the speed advantage enables him to forecast quote updates due to news arrival. Models of informed trading focus on the former type of advantage (accuracy) but not on the latter (speed).²

Our central finding is that the optimal trading strategy of the informed investor is very different when he has a speed advantage versus when he does not, holding the precision of his private information constant. In particular, a small speed advantage

¹News exploited by these traders are very diverse and include market events (quote updates, trades, orders), blog posts, news headlines, discussions in social forums etc. For instance, Brogaard, Hendershott, and Riordan (2012) show that high frequency traders in their data react to information contained in macro-economic announcements, limit order book updates, and market-wide returns. Data vendors such as Bloomberg, Dow-Jones or Thomson Reuters have started providing pre-processed real-time news feed to high frequency traders. For instance, in their on-line advertisement for real-time data processing tools, Dow Jones states: "Timing is everything and to make lucrative, well-timed trades, institutional and electronic traders need accurate real-time news available, including company financials, earnings, economic indicators, taxation and regulation shifts. Dow Jones is the leader in providing high-frequency trading." See http://www.dowjones.com/info/HighFrequencyTrading.asp.

²This is also the case for models that specifically analyze informed trading around news releases. For instance, Kim and Verrecchia (1994) assume that when news are released about the payoff of an asset, some traders ("information processors") are better able to interpret their informational content than market makers. As a result these traders have more accurate forecasts than market makers but receive news at the same time as other traders.

for the informed investor makes his optimal portfolio much more volatile, that is, the informed investor trades much more when he can react to news faster than market makers.

In our set-up, the informed investor has two motivations for trading. First, his forecast of the asset liquidation value is more precise than that of market makers. Second, by receiving news a split second before market makers, the informed investor can forecast market makers' quote updates due to public information arrival, that is, price changes in the very short run. The investor's optimal position in the risky asset reflects these two motivations: (i) its drift is proportional to market makers' forecast error (the difference between the informed investor's and market makers' estimates of the asset payoff) while (ii) its instantaneous variance is proportional to news. The second component (henceforth the *news trading component*) arises only if the informed investor has a speed advantage.³ The investor's position is therefore much more volatile in this case. Figure 1 illustrates this claim for one particular realization of news in our model.

This finding has several important and new implications. For instance, the informed investor's share of trading volume is much higher when he has a speed advantage. Indeed, the volatility of his order flow is of the same order of magnitude as the volatility of noise traders' order flow. Moreover, with a speed advantage, the informed investor's order flow at the high frequency (over a very short interval) has a positive correlation with subsequent returns, because the informed investor's trades are mainly driven by news arrivals, at high frequency. These features fit well with some stylized facts about high frequency traders: (a) their trades account for a large fraction of the trading volume (see Hendershott, Jones, and Menkveld (2011), Brogaard (2011), Brogaard, Hendershott, and Riordan (2012) or Chaboud, Chiquoine, Hjalmarsson, and Vega (2009)) and (b) their aggressive orders (i.e., marketable orders) anticipate very short run price changes (see Kirilenko, Kyle, Samadi, and Tuzun (2011) or Brogaard, Hendershott, and Riordan (2012)).⁴ In contrast, we show that the model in which the informed investor has more accurate information, but no speed advantage, cannot explain these facts.

Moreover, the effect of the precision of public information (that is, the news received

³In contrast, the drift of the investor's position is proportional to market makers' forecast error even when the investor has no speed advantage, as in the continuous time version of Kyle (1985) or extensions of this model such as Back (1992), Back and Pedersen (1998), Back, Cao, and Willard (2000) or Chau and Vayanos (2007).

⁴For instance, Kirilenko, Kyle, Samadi, and Tuzun (2011) note (page 21) that "possibly due to their speed advantage or superior ability to predict price changes, HFTs are able to buy right as the prices are about to increase."

Figure 1: Informed participation rate at various trading frequencies. The figure plots the evolution of the informed investor's position (upper panel) and the change in this position-the informed investor's trade- (lower panel) when the informed investor has a speed advantage (plain line) and when he has no speed advantage (dashed line) using the characterization of the optimal trading strategy for the investor in each case.



Informed Trading in Fast vs. Benchmark Model

by market makers) differs from that obtained in other models of trading around news, such as Kim and Verrecchia (1994). Usually, more precise public information is associated with greater market liquidity (lower price impact) but lower trading volume (see Kim and Verrecchia (1994) for instance). In contrast, in our model, it is associated with both an increase in liquidity (as market makers are less exposed to adverse selection), more trading volume, and a greater participation rate of the informed investor. Indeed, an increase in the precision of public information enables the informed investor to better forecast short run quote updates by market makers, which induces him to trade more aggressively on news. As a result, the volatility of his position increases, which means that both the trading volume and the fraction of this trading volume due to the informed investor increases. These effects imply that market makers are more exposed to adverse selection due to news trading but this effect is second order relative to the fact that they can better forecast the final payoff of the asset, so that they are less at risk of accumulating a long position when the asset liquidation value is low or vice versa. As a result, liquidity improves when public news are more precise, even though informed trading is more intense.⁵

The informed investor's ability to forecast quote updates also implies that short run returns are positively related to his contemporaneous order flow. This is indeed what Brogaard, Hendershott and Riordan (2012) find empirically for the aggressive trades of high frequency traders. In addition, they find that the same order flow of high frequency traders is negatively correlated with pricing errors. This is also the case in our model, as the informed investor sells on average when the price is above his estimate of the fundamental value and buys otherwise. But in our model, this behavior arises from the informed trader having market power. Indeed, the order flow of the informed trader moves the price by less than the innovation in asset value (the news), and thus his order flow has a negative contemporaneous correlation with the pricing error.

Last, we use the model to analyze the effects of speed on liquidity, price discovery, and volatility. This is of interest since speed is often viewed as the distinctive advantage of high frequency traders and the debate on high frequency trading revolves around

⁵This finding suggests that controlling for the precision of public information is important in analyzing the impact of high frequency news trading activity on liquidity. Indeed, when public information is more precise, both the informed investor's share of trading volume and liquidity improves. Thus, variations in the precision of public information across stocks or over time should work to create a positive association between liquidity and measures of high frequency news traders activity. Yet, this association is spurious since as explained below granting a speed advantage to the informed investor always impairs liquidity in our model.

the question of what is the effect of speed on measures of market performance (see for instance SEC (2010) or Gai, Yao, and Ye (2012)). To speak to this debate, we compare standard measures of market performance when the informed investor has a speed advantage (the new environment with HFTNs) and when he has not (the old environment without HFTNs) in our model.

Illiquidity (price impact of trades) is higher when the informed investor has a speed advantage because the ability of the informed investor to react faster to news is an additional source of adverse selection for market makers.⁶ Less obviously, this speed advantage also affects the nature of price discovery: price changes over short horizon are more correlated with innovations in the asset value (as found empirically in Brogaard, Hendershott, and Riordan (2012) but less correlated with the long run estimate of this value by the informed investor. The first effect improves price discovery while the second impairs price discovery. In equilibrium, they exactly cancel out so that the average pricing error (the difference between the transaction price and the informed investor's estimate of the asset value) is the same whether the informed investor has a speed advantage or not. Similarly, high frequency news trading alters the relative influences of trades and news arrivals on short run volatility. Trades move market makers' price more when the investor has a speed advantage because they are more informative about imminent news. But precisely for this reason, market makers' quotes are less sensitive to news because news have been partly revealed through trading. Therefore, the magnitude of quote revisions after news is smaller when the informed investor has a speed advantage, which dampens volatility. These two effect exactly offset each other so that overall high frequency news trading has no effect on volatility.

High frequency traders' strategies are heterogeneous (see SEC (2010)). Accordingly, they do not necessarily have all the same effects on market quality. In particular, some HFTs implicitly act as market makers (see Brogaard, Hendershott and Riordan (2012) or Menkveld (2012)). Market makers may use speed to protect themselves against better informed traders (e.g., by cancelling their limit orders just before news arrival) and provide liquidity at lower cost (see Jovanovic and Menkveld (2011)). This type of strategy is not captured by our model, which restricts the informed investor to submit market orders, as in Kyle (1985). This assumption is reasonable since Brogaard, Hendershott

 $^{^{6}}$ In line with this prediction, Hendershott and Moulton (2011) find that a reduction in the speed of execution for market orders submitted to the NYSE in 2006 is associated with larger bid-ask spreads, due to an increase in adverse selection.

and Riordan (2012) show empirically that only aggressive orders (i.e., market orders) submitted by high frequency traders are a source of adverse selection. However, it limits the scope of our implications. Accordingly, we do not claim that these implications are valid for all activities by high frequency traders.⁷

Our paper is related to the growing theoretical literature on high frequency trading.⁸ Our analysis is most related to Biais, Foucault, Moinas (2011) and Jovanovic and Menkveld (2011) who also build upon the idea that high frequency traders have a speed advantage in getting access to information. These models are static. Therefore they do not analyze the optimal dynamic trading strategy of an investor with fast access to news, while this analysis is central to our paper. Our approach is helpful to understand dynamic relationships between returns and the order flow of high frequency traders. Our framework does not lend itself to welfare analysis since it relies on the existence of noise traders. For a paper that discusses welfare issues and the social value of high frequency trading, see Biais, Foucault, and Moinas (2011).

Technically, our model is related to Back and Pedersen (1998) (BP(1998)), Chau and Vayanos (2008) (CV(2008)), and Martinez and Roşu (2012) (MR(2012)). As in BP(1998), one investor receives a continuous flow of information ("news") on the final payoff of an asset (its fundamental value) and optimally trades with market makers. As in CV(2008), market makers receives news continuously as well, but not as precisely as the investor.⁹ In contrast to both models, we assume that the informed investor observes news an infinitesimal amount of time before market makers. This feature implies that the instantaneous variance of the informed investor's position becomes strictly positive. MR(2012) obtains a similar finding for a different reason. In their model, market makers receive no news. In this particular case, the news trading component would disappear in our model. This is not the case in MR(2012) because the informed investor dislikes speculating on the long run value of the asset because of ambiguity aversion.

The paper is organized as follows. Section 2 describes our two models: the bench-

⁷This caveat is important for the interpretation of empirical findings in light of our predictions. For instance, Hasbrouck and Saar (2012) find a negative effect of their proxy for high frequency trading on volatility and a positive effect on liquidity while our model predicts respectively no effect and a negative effect of HFTNs on these variables. However, Hasbrouck and Saar (2012)'s proxy does not specifically capture the high frequency trades triggered by the arrival of news. Thus, it may be a noisy proxy for the trades of HFTNs.

⁸See, for instance, Cvitanic and Kirilenko (2011), Jovanovic and Menkveld (2011), Biais, Foucault, and Moinas (2011), Pagnotta and Philippon (2011), Cartea and Penalva (2012), or Hoffmann (2012).

 $^{^{9}}$ We take the greater precision of information for the investor as given. As in Kim and Verrecchia (1994), it could stem from greater processing ability for the informed investor.

mark model, and the fast model. Section 3 describes the resulting equilibrium price process and trading strategies, and compares the various coefficients involved. Section 4 discusses empirical implications of the model. Section 5 concludes. All proofs are in Appendix A. The model is set in continuous time, but in Appendix B we analyze the corresponding discrete time version. The goal of this analysis is to show that that the continuous time model captures the effects obtained in a discrete time model in which news and trading decisions are very frequent.

2 Model

Trading for a risky asset occurs over the time interval [0, 1]. The liquidation value of the asset at time 1 is v_1 . The risk-free rate is taken to be zero. Over the time interval [0, 1], a single informed trader ("he") and uninformed noise traders submit market orders to a competitive market maker ("she"), who sets the price at which the trading takes place. The informed trader learns about the asset liquidation value, v_1 , over time. His expectation of v_1 conditional on his information available until time t is denoted v_t . We refer to this estimate as the fundamental value of the asset at date t. This value follows a Gaussian process given by

$$v_t = v_0 + \int_0^t \mathrm{d}v_\tau, \quad \text{with} \quad \mathrm{d}v_t = \sigma_v \,\mathrm{d}B_t^v, \tag{1}$$

where v_0 is normally distributed with mean 0 and variance Σ_0 , and B_t^v is a Brownian motion.¹⁰ The informed trader observes v_0 at time 0, and observes dv_t during each time interval (t, t + dt], $t \in (0, 1)$. We refer to this innovation in asset value as the news received by the informed trader at t.

The position of the informed trader in the risky asset at t is denoted by x_t . As the informed trader is risk-neutral, he chooses x_t (his "trading strategy") to maximize his

¹⁰This assumption can be justified as follows. First, define the asset value v_t as the full information price of the asset, i.e., the price that would prevail at t if all information until t were to become public. Then, v_t moves any time there is news, which should be interpreted not just as information from newswires, but more broadly as changes in other correlated prices or economic variables such as trades in other securities etc. For example, Brogaard, Hendershott, and Riordan (2012), Jovanovic and Menkveld (2011) and Zhang (2012) show that the order flow of HFTs is correlated with changes in market-wide prices. Under this interpretation, v_t changes at a very high frequency, and can be assumed to be a continuous martingale, thus can be represented as an integral with respect to a Brownian motion (see the martingale representation theorem 3.4.2 in Karatzas and Shreve (1991)). Our representation (1) is then a simple particular case, with zero drift and constant volatility.

expected profit at t = 0 given by

$$U_0 = \mathsf{E}\left[\int_0^1 (v_1 - p_{t+dt}) \, \mathrm{d}x_t\right] = \mathsf{E}\left[\int_0^1 (v_1 - p_t - \mathrm{d}p_t) \, \mathrm{d}x_t\right],\tag{2}$$

where $p_{t+dt} = p_t + dp_t$ is the price at which the informed trader's order dx_t is executed.¹¹

The aggregate position of the noise traders at t is denoted by u_t . It follows an exogenous Gaussian process given by

$$u_t = u_0 + \int_0^t \mathrm{d}u_\tau, \quad \text{with} \quad \mathrm{d}u_t = \sigma_v \,\mathrm{d}B_t^u, \tag{3}$$

where B_t^u is a Brownian motion independent from B_t^v .

The market maker also learns about the asset value from (a) public information and (b) trades. During (t, t + dt], she receives a noisy signal of the innovation in asset value:

$$dz_t = dv_t + de_t, \quad \text{with} \quad de_t = \sigma_e \, dB_t^e, \tag{4}$$

where B_t^e is a Brownian motion independent from all the others. We refer to dz_t as the flow of news received by the market maker at date t. Furthermore, the market maker learns information from the aggregate order flow:

$$\mathrm{d}y_t = \mathrm{d}u_t + \mathrm{d}x_t,\tag{5}$$

because dx_t will reflect the information possessed by the informed trader (see below). We denote by q_t the market maker's expectation of the asset liquidation value just before she observes the aggregate order flow dy_t . As the market maker is competitive and riskneutral, she executes the order flow at a price equal to her expectation of the asset value just after she receives the order flow (as in Kyle (1985), BP (1998) or CV(2007)). We denote this transaction price by p_{t+dt} . As in Kyle (1985), one can interpret q_t as the bid-ask midpoint just before the transaction over (t, t + dt].¹²

If $\sigma_e > 0$, the news received by the market maker are less precise than those received by the informed trader. Thus, one advantage of the informed investor over the market

¹¹Because the optimal trading strategy of the informed trader might have a stochastic component, we cannot set $\mathsf{E}(\mathrm{d}p_t\mathrm{d}x_t) = 0$ as, e.g., in the Kyle (1985) model.

¹²This interpretation is correct if the price impact is increasing in the signed order flow and a zero order flow has zero price impact. These conditions are satisfied in the linear equilibrium we consider in Section 3.

Informed trader	Market maker's	Order	Execution	,
receives signai	quote	now	price	
$\mathrm{d}v_t$	q_t	$\mathrm{d}x_t + \mathrm{d}u_t$	$p_{t+\mathrm{d}t}$	
In benchmark:				In fast model:
Market maker				Market-maker
receives signal				receives signal
$\mathrm{d}z_t$				$\mathrm{d}z_t$

Figure 2: Timing of events during (t, t + dt] in the benchmark and the fast model

maker is that he can form a more precise forecast of the asset payoff than the market maker, at any point in time. As in Kim and Verrecchia (1994), this advantage could stem from the fact that the informed investor is better able to process news than the market makers.

Our focus here is on the second advantage for the informed investor: the possibility to trade on news faster than the market maker. To analyze this speed advantage and isolate its effects, we consider two different models: the *benchmark model* and the *fast model*. They differ in the timing with which the informed investor and the market maker receive news. The sequence of information arrival, quotes and trades in each model is summarized in Figure 2.

In the benchmark model, the order of events during the time interval (t, t + dt] is as follows. First, the informed trader observes dv_t and the market maker receives the signal dz_t . The market maker sets her quote q_t based on her information set $\mathcal{I}_t \cup dz_t$, where $\mathcal{I}_t \equiv \{z_\tau\}_{\tau \leq t} \cup \{y_\tau\}_{\tau \leq t}$, which comprises the order flow and the market maker's signals until time t, and the news just received in the interval (t, t + dt]. Then, the informed trader and the noise traders submit their market orders and the aggregate order flow, $dy_t = dx_t + du_t$ is realized. The information set of the market maker when she sets the execution price p_{t+dt} is therefore $\mathcal{I}_t \cup dz_t \cup dy_t$. That is, p_{t+dt} differs from q_t because it reflects the information contained in the order flow at date t + dt.

In the fast model, the informed trader can trade on news faster than the market maker. Namely, when the market maker executes the order flow dy_t , she does not yet observe the news dz_t while the informed investor has already observed the innovation in the asset value, dv_t . More specifically, over the interval (t, t + dt], the informed trader first observes dv_t , submits his market order dx_t along with the noise traders' orders du_t and the market maker executes the aggregate order flow at price p_{t+dt} , which is her conditional expectation of the asset payoff on the information set $\mathcal{I}_t \cup dy_t$. After trading has taken place and before the next trade, the market maker receives the signal dz_t and updates her estimate of the asset payoff based on the information set $\mathcal{I}_t \cup dz_t \cup dy_t$. Thus, the mid-quote q_{t+dt} at the beginning of the next trading round is the market maker's expectation of the asset payoff conditional on $\mathcal{I}_t \cup dz_t \cup dy_t$.

To sum up, in the benchmark model:

$$q_t = \mathsf{E}[v_1 \mid \mathcal{I}_t \cup \mathrm{d}z_t] \quad \text{and} \quad p_{t+\mathrm{d}t} = \mathsf{E}[v_1 \mid \mathcal{I}_t \cup \mathrm{d}z_t \cup \mathrm{d}y_t], \quad (6)$$

while in the fast model:

$$q_t = \mathsf{E}[v_1 \mid \mathcal{I}_t] \quad \text{and} \quad p_{t+\mathrm{d}t} = \mathsf{E}[v_1 \mid \mathcal{I}_t \cup \mathrm{d}y_t].$$
(7)

Thus, in the benchmark model, the market maker and the informed investor observe news (innovations in the asset value) at the same speed but not with the same precision (unless $\sigma_e = 0$). This information structure is standard in models of informed trading following Kyle (1985) and also in empirical applications (see Hasbrouck (1991a)). By contrast, in the fast model, the informed trader observes news a split second before the market maker. Thus, he also has a speed advantage relative to the market maker. Otherwise the benchmark model and the fast model are identical. Hence, by contrasting the properties of the benchmark model and the fast model, we can isolate the effects of high frequency traders' ability to react to news relatively faster than other market participants.

3 Optimal News Trading

In this section, we first derive the equilibrium of the benchmark model and the fast model. We then use the characterization of the equilibrium in each case to compare the properties of the informed investor's trades in each case.

3.1 Equilibrium

The equilibrium concept is similar to that of Kyle (1985) or Back and Pedersen (1998). That is, (a) the informed investor's trading strategy is optimal given market makers' pricing policy and (b) market makers' pricing policy follows equations (6) or (7) (depending on the model) with $dy_t = du_t + dx_t^*$ where dx_t^* is the optimal trading strategy for the informed investor. As usual in the literature using the framework of Kyle (1985), we look for equilibria in which prices are linear functions of the order flow and the informed investor's optimal trading strategy at date t (dx_t) is a linear function of his forecast of the asset value and the news he receives at date t.

More specifically, in the benchmark model, we look for an equilibrium in which the market maker's quote revision is linear in the public information she receives while the price impact is linear in the order flow. That is,

$$q_t = p_t + \mu_t^B \mathrm{d}z_t$$
 and $p_{t+\mathrm{d}t} = q_t + \lambda_t^B \mathrm{d}y_t$, (8)

where index B denotes a coefficient in the Benchmark case. In the fast model, we look for an equilibrium in which the transaction price, p_{t+dt} , is linear in the order flow as in equation (8) and the subsequent quote revision is linear in the *unexpected part* of the market maker's news. That is,

$$p_{t+\mathrm{d}t} = q_t + \lambda_t^F \mathrm{d}y_t \qquad \text{and} \qquad q_{t+\mathrm{d}t} = p_{t+\mathrm{d}t} + \mu_t^F (\mathrm{d}z_t - \rho_t^F \mathrm{d}y_t), \tag{9}$$

where $\rho_t^F dy_t$ is the market maker's expectation of the public information arriving over (t, t + dt] conditional on the order flow over this period and index F refers to the value of a coefficient in the Fast model. In the fast model, $\rho_t^F > 0$ because, as shown below, the informed investor's optimal trade at date t depends on the news received at this date (dv_t) . Thus, the market maker can forecast news from the order flow.

In both the benchmark and the fast model, we look for an equilibrium in which the informed investor's trading strategy is of the form

$$dx_t = \beta_t^k (v_t - q_t) dt + \gamma_t^k dv_t \quad \text{for } k \in \{B, F\}.$$
(10)

That is, we solve for β_t^k and γ_t^k so that the strategy defined in equation (10) maximizes the informed trader's expected profit (2). More generally, one may look for linear equilibria in which $dx_t = \int_0^t \gamma_j^k dv_j + \alpha_t$. However, we show in Appendix B that the optimal trading strategy for the informed investor in the discrete time version of our model is necessarily as in equation (10) when the market maker's pricing rule is linear. It is therefore natural to restrict our attention to this type of strategy in the continuous time version of the model.

The trading strategy of the informed investor at, say, date t has two components. The first component $(\beta_t(v_t - p_t)dt)$ is proportional to the market maker's forecast error, i.e., the difference between the forecast of the asset value by the informed investor and the forecast of this value by the market maker prior to the trade over (t, t + dt]. Intuitively, the informed investor buys when the market maker underestimates the fundamental value and sells otherwise. This component is standard in models of trading with asymmetric information such as Kyle (1985), Back and Pedersen (1998), Back, Cao, and Willard (2000), etc. In what follows, we refer to this component as being the forecast error component.

The second component of the informed investor's trading strategy is proportional to the news he receives at date t. We call it the *news trading* component. The next theorem shows that, in equilibrium, the news trading component is zero in the benchmark case $(\gamma_t^B = 0)$ while it is strictly positive in the case in which the informed investor has a speed advantage in reacting to news $(\gamma_t^F = 0)$. As explained in details below (see section 3.2), this difference implies that the informed investor's trades have very different properties when he is fast and when he is not. More generally, Theorem 1 provides a characterization of the equilibrium (coefficients μ_t^k , λ_t^k , ρ_t^F , β_t^k , and γ_t^k) in both the benchmark and the fast cases.

Theorem 1. In the benchmark model there is a unique linear equilibrium, of the form

$$dx_t = \beta_t^B (v_t - p_t) dt + \gamma^B dv_t, \qquad (11)$$

$$\mathrm{d}p_t = \mu^B \mathrm{d}z_t + \lambda^B \mathrm{d}y_t, \tag{12}$$

with coefficients given by

$$\beta_t^B = \frac{1}{1-t} \frac{\sigma_u}{\Sigma_0^{1/2}} \left(1 + \frac{\sigma_v^2 \sigma_e^2}{\Sigma_0(\sigma_v^2 + \sigma_e^2)} \right)^{1/2}, \tag{13}$$

$$\gamma^B = 0, \tag{14}$$

$$\lambda^{B} = \frac{\Sigma_{0}^{1/2}}{\sigma_{u}} \left(1 + \frac{\sigma_{v}^{2} \sigma_{e}^{2}}{\Sigma_{0} (\sigma_{v}^{2} + \sigma_{e}^{2})} \right)^{1/2},$$
(15)

$$\mu^B = \frac{\sigma_v^2}{\sigma_v^2 + \sigma_e^2}.$$
(16)

In the fast model there is a unique linear equilibrium, of the form:¹³

$$dx_t = \beta_t^F (v_t - q_t) dt + \gamma^F dv_t, \qquad (17)$$

$$dq_t = \lambda^F dy_t + \mu^F (dz_t - \rho^F dy_t), \qquad (18)$$

with coefficients given by

$$\beta_t^F = \frac{1}{1-t} \frac{\sigma_u}{(\Sigma_0 + \sigma_v^2)^{1/2}} \frac{1}{\left(1 + \frac{\sigma_e^2}{\sigma_v^2}g\right)^{1/2}} \left(1 + \frac{(1-g)\sigma_v^2}{\Sigma_0} \frac{1 + \frac{\sigma_e^2}{\sigma_v^2} + \frac{\sigma_e^2}{\sigma_v^2}g}{2 + \frac{\sigma_e^2}{\sigma_v^2} + \frac{\sigma_e^2}{\sigma_v^2}g}\right), \quad (19)$$

$$\gamma^{F} = \frac{\sigma_{u}}{\sigma_{v}} g^{1/2} = \frac{\sigma_{u}}{(\Sigma_{0} + \sigma_{v}^{2})^{1/2}} \frac{\left(1 + \frac{\sigma_{e}^{2}}{\sigma_{v}^{2}}g\right)^{1/2}(1+g)}{2 + \frac{\sigma_{e}^{2}}{\sigma_{v}^{2}} + \frac{\sigma_{e}^{2}}{\sigma_{v}^{2}}g},$$
(20)

$$\lambda^{F} = \frac{(\Sigma_{0} + \sigma_{v}^{2})^{1/2}}{\sigma_{u}} \frac{1}{\left(1 + \frac{\sigma_{e}^{2}}{\sigma_{v}^{2}}g\right)^{1/2}(1+g)},$$
(21)

$$\mu^{F} = \frac{1+g}{2 + \frac{\sigma_{e}^{2}}{\sigma_{v}^{2}} + \frac{\sigma_{e}^{2}}{\sigma_{v}^{2}}g},$$
(22)

$$\rho^{F} = \frac{\sigma_{v}}{\sigma_{u}} \frac{g^{1/2}}{1+g} = \frac{\sigma_{v}^{2}}{\sigma_{u}(\Sigma_{0} + \sigma_{v}^{2})^{1/2}} \frac{(1 + \frac{\sigma_{e}^{2}}{\sigma_{v}^{2}}g)^{1/2}}{2 + \frac{\sigma_{e}^{2}}{\sigma_{v}^{2}} + \frac{\sigma_{e}^{2}}{\sigma_{v}^{2}}g},$$
(23)

and g is the unique root in (0,1) of the cubic equation

$$g = \frac{\left(1 + \frac{\sigma_e^2}{\sigma_v^2}g\right)(1+g)^2}{\left(2 + \frac{\sigma_e^2}{\sigma_v^2} + \frac{\sigma_e^2}{\sigma_v^2}g\right)^2} \frac{\sigma_v^2}{\sigma_v^2 + \Sigma_0}.$$
 (24)

In both models, when $\sigma_v \to 0$, the equilibrium converges to the unique linear equilibrium in the continuous time version of Kyle (1985).

¹³Note that the forecast error component in (17) has q_t instead of p_t . This is the same formula, since (9) implies $(p_t - q_t)dt = 0$. We use q_t as a state variable, because p_t is not an Itô process.

The news trading component of the informed investor is non zero only if he has a speed advantage (and $\sigma_e < +\infty$ and $\sigma_v > 0$; see below). The reason for this important difference between the fast model and the benchmark model is as follows. In the fast model, the informed investor observes news an instant before the market maker. Thus, as long as $\sigma_e < +\infty$, he can forecast how the market maker will adjust her quotes in the very short run (equation (9) describes this adjustment) and trades on this knowledge, that is, buy just before an increase in price due to good news ($dv_t > 0$) or sell just before a decrease in prices due to bad news ($dv_t < 0$). As a result, $\gamma^F > 0$ if $\sigma_e < +\infty$. In contrast, in the benchmark case, the market maker incorporates news in her quotes before executing the informed investor's trade. As a result, the latter cannot exploit any very short-run predictability in prices and, for this reason, $\gamma^B = 0$.

Whether he is fast or not, the informed investor can form a forecast of the long run value of the asset, v_1 , that is more precise than that of the market maker both because he starts with an informational advantage (he knows v_0) and because he receives more informative news (if $\sigma_e > 0$). The informed investor therefore also exploits the market maker's pricing (or forecast) error, $v_t - q_t$. As usual, the trading strategy exploiting this advantage is to buy the asset when the market maker's pricing error is positive: $v_t - q_t > 0$ and to sell it otherwise. For this reason, the forecast error component of the strategy is strictly present whether the informed investor has a speed advantage or not $(\beta_t^k > 0 \text{ for } k \in \{B, F\}).$

Interestingly, the two components of the strategy can dictate trades in opposite directions. For instance, the forecast error component may call for additional purchases of the asset (because $v_t - q_t > 0$) when the news trading component calls for selling it (because $dv_t < 0$). The net direction of the informed investor's trade is determined by the sum of these two desired trades. Moreover, if the investor delegates the implementation of the two components of his trading strategy to two different agents (trading desks), one may see trades in opposite directions for these agents. Yet, they are part of an optimal trading strategy. Also, the two strategies cannot be considered independently in the sense that the sensitivity of the investor's trading strategy to the market maker's forecast error is optimally smaller when he has a speed advantage, as shown by the next proposition.

Proposition 1. For all values of the parameters and at each date: $\beta_t^F < \beta_t^B$.

Thus, in the fast model, the informed investor always exploits less aggressively the market maker's pricing error than in the benchmark case. In a sense, he substitutes profits from this source with profits from trading on news. The intuition for this *sub-stitution effect* is that trading more on news now reduces future profits from trading on the market maker's forecast error. Therefore, the informed investor optimally reduces the size of the trade exploiting the market maker's forecast error when he starts trading on news. As explained in Section 4, this substitution effect has an impact on the nature of price discovery. The next proposition describes how the sensitivities of the informed investor's trades to the market maker's forecast error and news vary with the exogenous parameters of the model.

Proposition 2. In the benchmark equilibrium and the fast equilibrium, β_t^B and β_t^F are increasing in σ_v , σ_u , σ_e , and decreasing in Σ_0 . Moreover, in the fast equilibrium, γ^F is increasing in σ_u , and decreasing in σ_e , Σ_0 .

An increase in σ_v or σ_u increases the informed investor's informational advantage. In the first case because news are more important (innovations in the asset value have a larger size) and in the second case because the order flow is noisier, other things equal. Thus, the informed investor reacts to an increase in these parameters by trading more aggressively on the market maker's forecast error.¹⁴

An increase in σ_e implies that the market maker receives noisier news. Accordingly, it becomes more difficult for the informed investor to forecast very short run price changes by the market maker. Hence, γ^F decreases with σ_e and goes to zero when σ_e goes to infinity. Thus, there is no news trading if the market makers do not receive news. Moreover, as the informed investor trades less aggressively on news, his trades become more sensitive to the market maker's forecast error (β_t^F increases) because of a substitution effect.¹⁵

When σ_e goes to $+\infty$, everything is as if the market maker never receives public information, as in Back and Pedersen (1998), since news for the market maker becomes uninformative. The equilibrium of the benchmark model in this case is identical to that obtained in Back and Pedersen (1998). If furthermore $\sigma_v = 0$, the informed investor

¹⁴The dependence on γ^F on σ_v is ambiguous, as when σ_v increases, γ^F first increases and then decreases, reflecting the fact that the price impact coefficient λ^F also increases with σ_v , which tempers the aggressiveness of the informed trader.

¹⁵Proposition 11 shows that the informed investor trades so that the total price informativeness is the same in both models.

receives no news and the benchmark case is then identical to the continuous time version of the Kyle (1985) model. In either case, the equilibrium of the fast model is identical to that of the benchmark case. In particular, even if the informed investor receives news faster than the market maker, his trading strategy will not feature a news trading component if the market maker does not receive news (γ^F goes to zero when σ_e goes to $+\infty$). Another polar case is the case in which $\sigma_e = 0$. In this case, the information contained in news is very short-lived for the informed investor. As implied by Proposition 2, the informed investor then trades very aggressively on news (γ^F is maximal when $\sigma_e = 0$).

3.2 The Trades of HFTNs

We now show that the behavior of the informed investor's order flow better coincides with stylized facts about high frequency traders when he has a speed advantage than when he has not.

The position of the informed investor, x_t , is a stochastic process. The drift of this process is equal to the forecast error component, while the volatility component of this process is determined by the news trading component. As the latter is zero in the benchmark case, the informed investor's trades at the high frequency (that is, the instantaneous change in the informed investor's position) are negligible relative to those of noise traders (they are of the order of dt while noise traders' trades are of the order of $(dt)^{1/2}$). In contrast, in the fast model, the informed investor's trades are of the same order of magnitude as those of noise traders, even at the high frequency. Thus, as shown on Figure 1, the position of the informed investor is much more volatile than in the benchmark case. Accordingly, over a short time interval, the fraction of total trading volume due to the informed investor is much higher when he has a speed advantage. To see this formally, let the *Informed Participation Rate* (IPR_t) be the contribution of the informed trader to total trading volume over an infinitesimal time interval (t, t + dt],

$$IPR_t = \frac{\mathsf{Var}(\mathrm{d}x_t)}{\mathsf{Var}(\mathrm{d}y_t)} = \frac{\mathsf{Var}(\mathrm{d}x_t)}{\mathsf{Var}(\mathrm{d}u_t) + \mathsf{Var}(\mathrm{d}x_t)}$$
(25)

Proposition 3. The informed participation rate is zero when the informed trader has no speed advantage, while it is strictly positive when he has a speed advantage:

$$IPR^B = 0, \qquad IPR^F = \frac{g}{1+g}, \tag{26}$$

where g is defined in Theorem 1.

The direction of the market maker's forecast error persists over time because the informed investor slowly exploits his private information (as in Kyle (1985) or Back and Pedersen (1998)). As a result, the forecast error component of the informed investor's trading strategy commands trades in the same direction for a relatively long period of time. This feature is a source of positive autocorrelation in the informed investor's order flow. However, when the informed investor has a speed advantage, over short time interval, trades exploiting the market maker's forecast error are negligible relative to those exploiting the short-run predictability in prices due to news arrival. As these trades have no serial correlation (since the innovations in asset value are not serially correlated), the autocorrelation of the informed order investor's order flow is smaller in the fast model. In fact the next result shows that over infinitesimal time intervals this autocorrelation is zero.

Proposition 4. Over short time intervals, the autocorrelation of the informed order flow is strictly positive when the informed investor has no speed advantage, and zero when the he has a speed advantage. For $\tau \in (0, 1 - t)$,

$$\operatorname{Corr}(\mathrm{d}x_t^B, \mathrm{d}x_{t+\tau}^B) = \left(\frac{1-t-\tau}{1-t}\right)^{\lambda^B \beta_0^B - \frac{1}{2}} > 0,$$

$$\operatorname{Corr}(\mathrm{d}x_t^F, \mathrm{d}x_{t+\tau}^F) = 0.$$
(27)

Proposition 3 and 4 hold when the order flow of the informed investor is measured over an infinitesimal time interval. Econometricians often work with aggregated trades over some time interval (e.g., 10 seconds), due to limited data availability or by choice, to make data analysis more manageable.¹⁶ In Appendix C, we show that the previous results are still qualitatively valid when the informed investor's trades are aggregated over time interval of arbitrary length (in this case, the informed investor's order flow over a given time interval is the sum of all of his trades over this time interval). In particular it is still the case that the informed investor's participation rate is higher while the autocorrelation of his order flow is smaller when he has a speed advantage. The only difference is that as flows are measured over longer time interval, the informed investor's participation rate in the benchmark as well as the autocorrelation of his trades

¹⁶For instance, Zhang (2012) aggregates the trades by HFTs in her sample over intervals of 10 seconds. However, trades in her sample happen at a higher frequency.

Figure 3: Informed participation rate at various sampling frequencies. The figure plots the fraction of the trading volume due to the informed trader when data are sampled over time intervals of various lengths (1 second, 1 minute, 1 hour, 1 day, 1 month) in (a) the benchmark model, marked with "*"; and (b) the fast model, marked with "o". The parameters used are $\sigma_u = \sigma_v = \sigma_e = \Sigma_0 = 1$ (see Theorem 1). The liquidation date t = 1 corresponds to 10 calendar years.



in the fast model both increase above zero. Indeed, the trades that the informed investor conducts to exploit the market maker's forecast error are positively autocorrelated and therefore account for an increasing fraction of his net order flow over longer time intervals. However, at relatively high sampling frequencies (e.g. daily), the participation rate of the informed investor remains low when he has no speed advantage, as shown on Figure 3. Thus, the model in which the informed investor has no speed advantage does not explain well why high frequency traders account for a large fraction of the trading volume.

Using US stock trading data aggregated across twenty-six HFTs, Brogaard (2011) finds a positive autocorrelation of the aggregate HFT order flow, which is consistent both with the benchmark model and the model in which the informed investor has a speed advantage, provided the sampling frequency is not too high. In addition, our model implies that this autocorrelation should decrease with the sampling frequency in the fast model (see Proposition 13 in Appendix C). In contrast, Menkveld (2011) using data on a single HFT in the European stock market, and Kirilenko, Kyle, Samadi, and Tuzun (2011) using data on the Flash Crash of May 2010, find evidence of mean reverting positions for HFTs. One possibility is that HFTNs face inventory constraints

due to risk management concerns. While this feature is absent from our model, such constraints would naturally lead to mean reversion in the informed investor's trades. Alternatively, these empirical studies may describe the behavior of a different category of high frequency traders we do not model, namely the high frequency market makers. Menkveld (2011) shows that the high frequency trader in his dataset behaves very much as a market maker rather than an informed investor.

Some empirical papers also find that aggressive orders by HFTs (that is, marketable orders) have a very short run positive correlation with subsequent returns (see Brogaard, Hendershott and Riordan (2012) and Kirilenko, Kyle, Samadi, and Tuzun (2011)). This finding is consistent with our model when the informed investor has a speed advantage but not otherwise. To see this, let AT_t (which stands for Anticipatory Trading) be the correlation between the informed order flow at a given date and the next instant return, that is:

$$AT_t = \mathsf{Corr}(\mathrm{d}x_t, q_{t+\mathrm{d}t} - p_{t+\mathrm{d}t}), \tag{28}$$

where we recall that p_{t+dt} is the price at which the trade dx_t is executed, and q_{t+dt} is the next quote posted by the market maker after she receives additional news (see Figure 2).

Proposition 5. Anticipatory trading is zero when the informed investor has no speed advantage, while it is strictly positive when he has a speed advantage:

$$AT^{B} = 0,$$

$$AT^{F} = \frac{1}{\sqrt{(1+g)(1+\frac{\sigma_{e}^{2}}{\sigma_{v}^{2}})}} > 0,$$
(29)

where $g \in (0, 1)$ is as in Theorem 1.

When the informed investor observes news an instant before the market maker, his order flow over a short period of time is mainly determined by the direction of incoming news. Thus, his trades anticipate on the adjustment of his quotes by the market maker, which creates a short run positive correlation between the trades of the informed investor and subsequent returns, as observed in reality.¹⁷

¹⁷Anticipatory trading in our model refers to the ability of the informed investor to trade ahead of incoming news. The term "anticipatory trading" is sometimes used to refer to trades ahead of or alongside other investors, for instance institutional investors (see Hirschey (2011)). This form of anticipatory trading is not captured by our model.

In Appendix C, we analyze how this result generalizes when the sampling frequency used by the econometrician is lower than the frequency at which the informed investor trades on news. We show (see Proposition 14 in Appendix C) that the correlation between the aggregate order flow of the informed investor over an interval of time of fixed length and the asset return over the next time interval (of equal length) declines when the frequency at which data are sampled decreases relative to the frequency at which the investor trades and goes to zero when the ratio of sampling frequency to trading frequency goes to zero (as in the continuous time model). Thus, the choice of a sampling frequency to study high frequency news trading is not innocuous and can affect inferences. If this frequency is too low relative to the frequency at which trades take place (which by definition is very high for high frequency traders), it would be more difficult to detect the presence of anticipatory trading by the informed investor.

4 Empirical Implications

4.1 News Informativeness, Volume and Liquidity

Empirical findings suggest that the activity of high frequency traders vary across stocks (e.g., Brogaard, Hendershott, and Riordan (2012) find that HFTs are more active in large cap stocks than small cap stocks). Our model suggests two possible important determinants of the activity of high frequency traders on news, measured by their participation rate as defined in equation (25): (i) the precision of the public information received by market makers and (ii) the informational content of the news received by the informed investor.

Following Kim and Verrecchia (1994), we measure the precision of public information by σ_e since a smaller σ_e means that the news received by the market maker provide a more precise signal about innovations in the asset value.¹⁸ Moreover we measure the volume of trading by $Var(dy_t)$, a measure of the average absolute order imbalance in each transaction.

Proposition 6. In the fast model, an increase in the precision of public news, i.e., a decrease in σ_e , results in (i) higher participation of the informed investor (IPR^F) , (ii) higher trading volume (Var(dy)), and (iii) higher liquidity (lower λ^F).

¹⁸Holding constant the variance of the innovation of the asset value σ_v^2 , more precise public news about the changes in asset value amounts to a lower $\sigma_z^2 = \sigma_v^2 + \sigma_e^2$, or, equivalently, a lower σ_e^2 .

When public information is more precise, the informed investor trades more aggressively on the news he receives as shown by Proposition 2. Indeed, the market maker's quotes are then more sensitive to news (μ^F decreases in σ_e) and, as a result, the informed investor can better exploit his foreknowledge of news when he receives news faster than the market maker. As a result, he trades more over short-time interval so that his participation rate and trading volume increase.

An increase in the precision of public information has an ambiguous effect on the exposure to adverse selection for the market maker. On the one hand, it increases the sensitivity of the informed investor's trade to news, which increases the exposure to adverse selection for the market maker. On the other hand, it helps the market maker to better forecast the asset liquidation value, which reduces his exposure to adverse selection. As shown by Proposition 6, the second effect always dominates so that illiquidity is reduced when the market maker receives more precise news.

These findings are in sharp contrast with other models analyzing the effects of public information or corporate disclosures, such as Kim and Verrecchia (1994). Indeed, in these models, an increase in the precision of public information leads to *less* trading volume (as informed investors trade less) and greater market liquidity. Furthermore, they suggest that controlling for the precision of public information is important to analyze the effect of high frequency trading on liquidity. Indeed, in our model, variations in the precision of public information lead to a positive association between liquidity and the activity of high frequency news traders, but this does not imply that high frequency news trading causes the market to be more liquid (instead, we show in Proposition 10 that the opposite is true).

To test the implications of Proposition 6, one needs a proxy for the precision of public news received by market makers. For this, one can consider various "news sentiment scores" that are provided by data vendors such as Reuters, Bloomberg, Dow Jones (see for instance Gross-Klussmann and Hautsch, 2011). These vendors report firm-specific news in real time and assign a direction to the news (a proxy for the sign of dz_t) and a relevance score to news. Thus, as proxy for σ_e , we suggest the average relevance score of news about a firm (or a portfolio of firms). Indeed, firms with more relevant news should be firms for which public information is more precise.

In our model, the informed investor has two sources of information: (i) his initial forecast v_0 and (ii) news about the asset value. His initial forecast is never disclosed

to market makers and can be seen as private information in the traditional sense. In contrast, the news that the informed investor receives are partially revealed to market makers and his corresponding information advantage is very short lived if σ_e is small. When σ_v increases, news become relatively more informative than private information for the informed investor since news account for a greater fraction of the total volatility of the liquidation value for the asset $\left(\frac{\sigma_v^2}{\sigma_v^2 + \Sigma_0}\right)$ increases in σ_v). At the same time, for a fixed value of σ_e , the news received by the market maker are less informative.¹⁹ Thus, to analyze the effect of increasing the informational content of news for the informed investor *ceteris paribus*, the next proposition considers the effect of a change in σ_v , while holding constant the ratio $\frac{\sigma_e}{\sigma_v}$.

Proposition 7. In the fast model, an increase in the informational content of news for the informed investor, i.e., an increase in σ_v holding fixed σ_e/σ_v , results in a (i) higher participation rate for the informed investor (IPR^{F}) , (ii) higher trading volume, and (iii) lower liquidity.

A greater value of σ_v increases the informed trader's profit from speed advantage. As a result the participation rate of the informed investor and trading volume increase. Simultaneously, the exposure to adverse selection for the market maker increases and therefore illiquidity increases. As a proxy for σ_v , we suggest the number of times the news sentiment score for a given firm exceeds a certain threshold over a fixed period of time (say the day). Intuitively, firms for which this number is high are firms for which more information is released over time (that is firms for which $\frac{\sigma_v^2}{\sigma_n^2 + \Sigma_0}$ is higher).

Last, an increase in σ_v is associated with a higher price volatility since $\frac{1}{dt} \operatorname{Var}(dp_t) =$ $\sigma_v^2 + \Sigma_0$. Thus, an increase in the informational content of news received by HFTNs leads to a positive association between the volatility of short-run return and the activity of high frequency traders on news, although this does not imply that HFTNs have no causal effect on volatility (in Proposition 12 below, we show that they have no effect on volatility).²⁰

¹⁹Indeed, $\frac{1}{dt} \operatorname{Var}(dv_t | dz_t) = \frac{\sigma_v^2 \sigma_e^2}{\sigma_v^2 + \sigma_e^2}$, which increases with σ_v . ²⁰In line with this prediction, Chaboud et al. (2010) find that high frequency traders in their data are more active on days with high volatility but do not appear to cause higher volatility.

4.2 Price Discovery and News Trading

Theorem 1 shows that the informed investor trades aggressively on news when he has a speed advantage. This should impact how the informed trader contributes to price discovery. To analyze this point more formally, we consider two econometric models that empiricists commonly use to estimate the informational content of trades: (i) the VAR model and (ii) the state space model.

Hasbrouck (1991a) advocates the use of Vector Autoregressive models to estimate the informational content of a trade while accounting for autocorrelation in trades and returns. Some researchers, e.g., Zhang (2012) or Hirschley (2012), therefore use this approach to measure the informational content of high frequency traders' order flows. In order to make our model more comparable to econometric models, we consider a discrete time version of our fast model, as described in Appendix B. It works very similarly to the continuous time model, the main difference being that the infinitesimal time interval dt is replaced by a real number $\Delta t > 0$. We consider Δt small, so that we approximate the equilibrium variables (β_t , γ_t , λ_t , μ_t , ρ_t) in the discrete time model by their continuous time counterpart. For simplicity, we also write t + 1 instead of $t + \Delta t$.

In the model with speed advantage, the informed order flow at t is (omit the superscript F):

$$\Delta x_t = \beta_t (v_t - q_t) \Delta t + \gamma \Delta v_t = \gamma \Delta v_t + O(\Delta t).$$
(30)

where q_t is the quote just before the trading at t. From Theorem 1, the order flow executes at $p_{t+1} = q_t + \lambda \Delta y_t = p_t + \mu(\Delta z_{t-1} - \rho \Delta y_{t-1}) + \lambda \Delta y_t$. If we denote by $r_t = p_{t+1} - p_t$ the return contemporaneous to Δx_t , we have the following orthogonal decomposition:

$$r_t = \lambda \Delta y_t + \mu (\Delta z_{t-1} - \rho \Delta y_{t-1}). \tag{31}$$

Now, using the fact that $\Delta y_t = \Delta x_t + \Delta u_t$ and the fact that $\Delta x_{t-1} = \gamma \Delta v_{t-1} + O(\Delta t)$, we obtain the following result.

Proposition 8. The model in which the informed investor has a speed advantage implies the following VAR model for short run returns and trades by the informed investor. Thus, up to terms of order $O(\Delta t)$,

$$r_{t} = \lambda \Delta x_{t} + \frac{\mu}{\gamma} (1 - \gamma \rho) \Delta x_{t-1} + \varepsilon_{t},$$

$$\Delta x_{t} = \eta_{t},$$
(32)

where

$$\varepsilon_t = \lambda \Delta u_t + \mu \Delta e_{t-1} - \mu \rho \Delta u_{t-1},$$

$$\eta_t = \gamma \Delta v_t.$$
(33)

Hasbrouck (1991a) proposes to measure the informational content of a trade at date t by the *permanent impact of a trade*, which is the sum of predicted quote revisions through a fixed number of steps after a trade innovation of a fixed size. In our case, after at least two steps, the permanent price impact of a trade is

$$\lambda_{\text{perm}} = \lambda + \frac{\mu}{\gamma} (1 - \gamma \rho) = \lambda + \frac{\mu}{\gamma} \frac{1}{1 + g} > \lambda, \qquad (34)$$

where $g \in (0, 1)$ is as in Theorem 1. This implies that the Hasbrouck measure λ_{perm} always overestimates λ , the true permanent impact of the informed investor's trade. Indeed, the lagged informed order flow Δx_{t-1} appears in equation (32) not because the market maker's quotes slowly adjust to information contained in past trades, but because Δx_{t-1} anticipates the news received by the market maker at t, i.e., Δx_{t-1} is correlated with the market maker's quote update $\mu \Delta z_{t-1}$.

This observation suggests that one must be careful in interpreting measures of the informational content of trades using the Hasbrouck (1991a)'s approach when informed investors trade on public information. This problem is in fact discussed by Hasbrouck (1991a) (see Section III in his paper). It may have become more severe in recent years with the development of high frequency trading on news.

Brogaard, Hendershott and Riordan (2012) use a state space model, developed by Menkveld, Koopman and Lucas (2007), to analyze the effect of high frequency traders on price formation. The price is decomposed into a permanent component w_t and a transitory component s_t . The permanent component is a martingale whose innovation is a function of the innovation in the informed investor's order flow: $\Delta w_t = \kappa \Delta \tilde{x}_t + \varepsilon_t$. The transitory component is a stationary autoregressive component whose innovation depends on the informed investor's trades: $s_{t+1} = \phi s_t + \psi \Delta x_t + \eta_t$.

In our model with speed advantage, the innovation of Δx_t is simply the news trading component $\gamma \Delta v_t$, since the forecast error component is predictable by the informed trader. Therefore, the permanent component is $w_t = v_t$ and its innovation is entirely explained by informed trades, i.e., $\kappa = \frac{1}{\gamma}$ and $\varepsilon_t = 0$. The transitory component is equal to the pricing error $s_t = q_t - v_t$ and, after rearranging the pricing equation (18), we prove the following result.²¹

Proposition 9. The model in which the informed investor has a speed advantage implies the following state model:

$$\Delta w_t = \kappa \Delta x_t + \varepsilon_t, \qquad s_{t+1} = \phi s_t + \psi \Delta x_t + \eta_t, \quad with$$

$$\kappa = \frac{1}{\gamma} > 0, \quad \phi = 1 - m\beta \Delta t, \quad \psi = -\frac{1}{\gamma}(1 - \mu - m\gamma) < 0,$$
(35)

where $m = \lambda - \mu \rho$, and the other coefficients are as in Theorem 1. (See the Appendix for more details.)

Brogaard, Hendershott and Riordan (2012) interpret a positive κ as HFTs contributing to the discovery of the efficient price. This interpretation is consistent with our model, because our news trading component is proportional to the innovation in asset value. In the transitory equation, a negative ψ is interpreted as the informed investor trading against pricing errors and reducing the noise in the price. Our model suggests that this interpretation should be taken cautiously since $\psi < 0$ does not come from the forecast error component of the trading strategy, i.e., β does not enter the equation for ψ . Instead, ψ is negative because of the news trading component. To see why, suppose for instance that the informed trader receives positive news ($\Delta v_t > 0$) and thus buys the asset $(\Delta x_t = \gamma \Delta v_t + O(\Delta t))$. This raises the price, but because the informed trader has market power, the price increase is smaller than the increase in asset value, i.e., $\lambda \gamma \Delta v_t < \Delta v_t$.²² As a result, there is a negative pricing error precisely at the moment when the informed trader is buying. In other words, the news trading component generates a negative correlation between trades and pricing errors. By contrast, the forecast error component of the trading strategy does not generate the same effect, because it is

²¹We consider the price to be q_t , the quote just before trading. If instead we define $s_t = p_t - v_t$ using the actual trading price, the state space model is also true, but the formulas are more complicated. ²²This is because $\lambda \gamma = \left(2 + \frac{\sigma_e^2}{\sigma_v^2}(1+g)\right)^{-1} < 1$. See also equation (83) in the Appendix.

actually included in the autoregressive component ϕs_{t-1} .

4.3 The Effect of HFTN on Market Quality

Controversies about high frequency traders focus on the effects of their speed advantage on liquidity, price discovery and price volatility. In this section, we study how the informed investor's ability to react to news faster than the market maker (i.e., the presence of high frequency trading on news) affects measures of market quality. To this end, we compare these measures when the informed investor has a speed advantage and when he has not, holding the precision of the informed investor's information constant.

As in Kyle (1985), we measure market illiquidity by λ , the immediate price impact of a trade.

Proposition 10. Liquidity is lower when the informed investor has a speed advantage, i.e., $\lambda^F > \lambda^B$.

Trades by the informed investor expose the market maker to adverse selection because the informed investor has a more accurate forecast of the asset liquidation value than the market maker. Thus, the market maker tends to accumulate a short position when she underestimates the asset value and a long position when she overestimates the asset value. This source of adverse selection is present both when the investor has a speed advantage and when he has not. However, adverse selection is stronger when the informed investor has a speed advantage because he can also buy just in advance of positive news and sell in advance of negative news. As a result, market illiquidity is higher when the informed investor has a speed advantage. This is consistent with Hendershott and Moulton (2011), who find that a speed reduction for market orders on the NYSE in 2006 is associated with less liquidity.

Next, we consider the effect of HFTN on price discovery. We measure price discovery by the average squared pricing error at t, i.e.,

$$\Sigma_t = \mathsf{E}((v_t - p_t)^2). \tag{36}$$

The smaller is Σ_t , the higher is informational efficiency. The change in Σ_t is given by²³

$$d\Sigma_t = -2\operatorname{Cov}(dp_t, v_t - p_t) - 2\operatorname{Cov}(dp_t, dv_t) + (2\sigma_v^2 + \Sigma_0)dt.$$
(37)

Thus, price discovery improves when short run changes in prices are more correlated with (a) news (i.e., $Cov(dp_t, dv_t)$ increases), and (b) the direction of the market maker's forecast error (i.e., $Cov(dp_t, v_t - p_t)$ increases). Interestingly, granting a speed advantage to the informed investor has opposite effects on the two dimensions of price discovery.

Proposition 11. When the informed investor has a speed advantage, short run changes in prices are more correlated with innovations in the asset value (i.e., $Cov(dp_t, dv_t)$ is higher), but less correlated with the market maker's forecast error (i.e., $Cov(dp_t, v_t - p_t)$ is smaller). Overall, Σ_t is identical whether or not the informed investor has a speed advantage.

Hence, the speed advantage of the informed investor does not increase or reduce pricing errors on average. However, it changes the *nature* of price discovery in the short run. In a nutshell, returns are more informative about the level of the asset value in the benchmark model, while they are more informative about changes in the asset value in the fast model. When the informed trader has a speed advantage, returns are more correlated with news because he trades aggressively on news. By contrast, returns are less correlated with the level of the asset value in the fast model, because the informed investor trades less aggressively on the market maker's forecast error when he has a speed advantage ($\beta_t^F < \beta_t^B$), as shown in Proposition 1.

In equilibrium, these two effects exactly cancel out so that eventually the pricing error is the same in both models. In the fast model, new information is incorporated more quickly into the price while older information is incorporated less quickly, leaving the total pricing error equal in both models.²⁴

We now consider the effect of HFTN on the volatility of short run returns, $Var(dp_t)$. This volatility has two sources in our model: (i) trading, and (ii) quote updates. The

²³We compute $d\Sigma_t = 2 \operatorname{Cov}(dv_t - dp_t, v_t - p_t) + \operatorname{Cov}(dv_t - dp_t, dv_t - dp_t)$. Since the news dv_t is orthogonal to $v_t - p_t$, $d\Sigma_t = -2 \operatorname{Cov}(dp_t, v_t - p_t) - 2 \operatorname{Cov}(dp_t, dv_t) + \operatorname{Var}(dv_t) + \operatorname{Var}(dp_t)$. But $\frac{1}{dt} \operatorname{Var}(dv_t) = \sigma_v^2$; and by Proposition 12, $\sigma_p^2 = \frac{1}{dt} \operatorname{Var}(dp_t) = \sigma_v^2 + \Sigma_0$.

²⁴More formally, the two effects exactly offset each other because, in both models the informed trader releases information at a constant rate, hence Σ_t decreases linearly over time. The transversality condition for optimization requires that no money is left on the table at t = 1, i.e., $\Sigma_1 = 0$. Since the initial value Σ_0 is exogenously given, the evolution of Σ_t is the same in both models.

second source of volatility is reflected into the quote adjustments due to the news received by the market maker. Thus, following Hasbrouck (1991b), we decompose price volatility into the volatility coming from trades and the volatility coming from quotes:

$$\mathsf{Var}(\mathrm{d}p_t) = \mathsf{Var}(p_{t+\mathrm{d}t} - q_t) + \mathsf{Var}(q_t - p_t).$$
(38)

Proposition 12. Whether the informed investor has a speed advantage or not, the instantaneous volatility of returns is constant, and equal to

$$\sigma_p^2 = \frac{1}{\mathrm{d}t} \operatorname{Var}(\mathrm{d}p_t) = \sigma_v^2 + \Sigma_0.$$
(39)

Trades contribute to a larger fraction of this volatility when the informed investor has a speed advantage.

Thus, the speed advantage of the informed investor alters the contribution of each source of volatility. In the fast model, trades contribute more to volatility since trades are more informative on impending news. The flip side is that the market maker's quote is less sensitive to news. Thus, the contribution of quote revisions to short run return volatility is lower in the fast model. These two effects cancel each other in equilibrium so that volatility is the same in both models.

5 Conclusion

Adverse selection occurs in financial markets because certain investors have either more precise information, or superior speed in accessing or exploiting information. To disentangle the effects of precision and speed on market performance, we have derived the optimal trading strategy of an informed investor when he reacts to news either (i) at the same speed, or (ii) faster than the other market participants, holding information precision constant.

Our main result is that the optimal trading strategy of the informed investor is very different when has a speed advantage versus when he does not. In general, the optimal trading strategy has two components: (i) the forecast error component, which is proportional to the difference between the informed investor's and market makers' estimates of the asset payoff; and (ii) the news trading component, which is proportional to the innovation in the asset value. We have shown that the news trading component occurs only when the informed investor has a speed advantage.

As a consequence, with a speed advantage for the informed investor, his optimal portfolio is much more volatile, because of the news trading component. Also, at very high frequencies his order flow is correlated with subsequent returns, because news trading is driven by news arrivals. These features fit well with some stylized facts about high frequency traders documented in the literature. For instance, the trades of HFTs account for a large fraction of the trading volume. Moreover, the marketable orders of HFTs anticipate very short run price changes. In contrast, we show that the model in which the informed investor has more accurate information, but no speed advantage, cannot explain these facts.

We have defined high frequency trading on news (HFTN) as trading by informed investors with a speed advantage, as in our model. We have two types of empirical predictions: (i) the effect of different market characteristics on HFTN activity; and (ii) the effect of HFTN on certain measures of market quality. As an example of prediction of type (i), we show that an increase in the precision of public news increases news trading, i.e. HFTN activity, yet surprisingly improves liquidity. For type (ii), we find that an increase in HFTN activity (a) increases trading volume; (b) does not affect total price volatility, and (c) increases overall adverse selection, and thus decreases market liquidity.

Our paper is related to a fast growing literature on high frequency trading. One caveat about interpreting our work is that HFTN can be identified only with a subcategory of high frequency trading, and not with all HFT. To draw more general conclusions, one should extend the model in several directions. First, investors' information could refer not only to the asset value, but also to the order flow of other traders. Second, inventory constraints can explain the mean reversion of inventories observed in practice. Third, allowing informed traders to also submit limit orders would extend the model to another important category of HFT, the high frequency market makers.

A Proofs of Results

A.1 Proof of Theorem 1

Benchmark model: We compute the optimal strategy of the informed trader at t+dt. As explained in the discussion before Theorem 1, we consider only strategies dx_{τ} of the type $dx_{\tau} = \beta_{\tau}^{B}(v_{\tau} - p_{\tau}) d\tau + \gamma_{\tau}^{B} dv_{\tau}$. Recall that \mathcal{I}_{t}^{p} is the market maker's information set immediately after trading at t. If we denote by $\mathcal{J}_{t}^{p} = \mathcal{I}_{t}^{p} \cup \{v_{\tau}\}_{\tau \leq t+dt}$ the trader's information set before trading at t + dt, the expected profit from trading after t is

$$\pi_t = \mathsf{E}\left(\int_t^1 (v_1 - p_{\tau + d\tau}) \, \mathrm{d}x_\tau \mid \mathcal{J}_t^p\right).$$
(40)

From (12), $p_{\tau+d\tau} = p_{\tau} + \mu_{\tau}^B (\mathrm{d}v_{\tau} + \mathrm{d}e_{\tau}) + \lambda_{\tau}^B (\mathrm{d}x_{\tau} + \mathrm{d}u_{\tau})$. For $\tau \ge t$, denote by

$$V_{\tau} = \mathsf{E}\big((v_{\tau} - p_{\tau})^2 \mid \mathcal{J}_t^p\big).$$
(41)

For convenience, we now omit the superscript B for the coefficients $\beta, \gamma, \mu, \lambda$. Then the expected profit is

$$\pi_t = \mathsf{E}\left(\int_t^1 (v_\tau + \mathrm{d}v_\tau - p_\tau - \mu_\tau \,\mathrm{d}v_\tau - \lambda_\tau \,\mathrm{d}x_\tau) \,\mathrm{d}x_\tau \mid \mathcal{J}_t^p\right) = \int_t^1 \left(\beta_\tau V_\tau + (1 - \mu_\tau - \lambda_\tau \gamma_\tau) \gamma_\tau \sigma_v^2\right) \mathrm{d}\tau.$$
(42)

 V_{τ} can be computed recursively:

$$V_{\tau+d\tau} = \mathsf{E} \left((v_{\tau+d\tau} - p_{\tau+d\tau})^2 \mid \mathcal{J}_t^p \right)$$

= $\mathsf{E} \left((v_{\tau} + dv_{\tau} - p_{\tau} - \mu_{\tau} dv_{\tau} - \mu_{\tau} de_{\tau} - \lambda_{\tau} dx_{\tau} - \lambda_{\tau} du_{\tau})^2 \mid \mathcal{J}_t^p \right)$ (43)
= $V_{\tau} + (1 - \mu_{\tau} - \lambda_{\tau} \gamma_{\tau})^2 \sigma_v^2 d\tau + \mu_{\tau}^2 \sigma_e^2 d\tau + \lambda_{\tau}^2 \sigma_u^2 d\tau - 2\lambda_t \beta_t V_{\tau} d\tau.$

therefore the law of motion of V_{τ} is a first order differential equation

$$V_{\tau}' = -2\lambda_t \beta_t V_{\tau} + (1 - \mu_{\tau} - \lambda_{\tau} \gamma_{\tau})^2 \sigma_v^2 + \mu_{\tau}^2 \sigma_e^2 + \lambda_{\tau}^2 \sigma_u^2,$$
(44)

or equivalently $\beta_{\tau}V_{\tau} = \frac{-V'_{\tau} + (1 - \mu_{\tau} - \lambda_{\tau}\gamma_{\tau})^2 \sigma_v^2 + \mu_{\tau}^2 \sigma_e^2 + \lambda_{\tau}^2 \sigma_u^2}{2\lambda_{\tau}}$. Substitute this into (40) and integrate by parts. Since $V_t = 0$, we get

$$\pi_t = -\frac{V_1}{2\lambda_1} + \int_t^1 V_\tau \left(\frac{1}{2\lambda_\tau}\right)' d\tau + \int_t^1 \left(\frac{(1-\mu_\tau - \lambda_\tau \gamma_\tau)^2 \sigma_v^2 + \mu_\tau^2 \sigma_e^2 + \lambda_\tau^2 \sigma_u^2}{2\lambda_\tau} + (1-\mu_\tau - \lambda_\tau \gamma_\tau) \gamma_\tau \sigma_v^2\right) d\tau.$$
(45)

This is essentially the argument of Kyle (1985): we have eliminated the choice variable β_{τ} and replaced it by V_{τ} . Since $V_{\tau} > 0$ can be arbitrarily chosen, in order to get an

optimum we must have $\left(\frac{1}{2\lambda_{\tau}}\right)' = 0$, which is equivalent to

$$\lambda_{\tau} = \text{constant} = \lambda. \tag{46}$$

For a maximum, the transversality condition $V_1 = 0$ must be also satisfied.

We next turn to the choice of γ_{τ} . The first order condition is

$$-(1 - \mu_{\tau} - \lambda_{\tau}\gamma_{\tau}) + (1 - \mu_{\tau} - \lambda_{\tau}\gamma_{\tau}) - \lambda_{\tau}\gamma_{\tau} = 0 \implies \gamma_{\tau} = 0.$$
(47)

Thus, there is no news trading in the benchmark model. Note also that the second order condition is $\lambda_{\tau} > 0.2^{5}$

Next, we derive the pricing rules from the market maker's zero profit conditions. The equations $p_t = \mathsf{E}(v_1 | \mathcal{I}_t^p)$ and $q_t = \mathsf{E}(v_1 | \mathcal{I}_t^p, \mathrm{d}z_t)$ imply that $q_t = p_t + \mu_t \,\mathrm{d}z_t$, where

$$\mu_t = \frac{\operatorname{Cov}(v_1, \, \mathrm{d}z_t \mid \mathcal{I}_t^p)}{\operatorname{Var}(\, \mathrm{d}z_t \mid \mathcal{I}_t^p)} = \frac{\operatorname{Cov}(v_0 + \int_0^1 \, \mathrm{d}v_\tau, \, \mathrm{d}v_t + \, \mathrm{d}e_t \mid \mathcal{I}_t^p)}{\operatorname{Var}(\, \mathrm{d}v_t + \, \mathrm{d}e_t \mid \mathcal{I}_t^p)} = \frac{\sigma_v^2}{\sigma_v^2 + \sigma_e^2} = \mu.$$
(48)

The equations $q_t = \mathsf{E}(v_1 | \mathcal{I}_{t+dt}^q)$ and $p_{t+dt} = \mathsf{E}(v_1 | \mathcal{I}_{t+dt}^q, \mathrm{d}y_t)$ imply that $p_{t+dt} = q_t + \lambda_t \mathrm{d}y_t$, where, since $\lambda_t = \lambda$ is constant,

$$\lambda = \frac{\operatorname{Cov}(v_1, \, \mathrm{d}y_t \mid \mathcal{I}_{t+\mathrm{d}t}^q)}{\operatorname{Var}(\, \mathrm{d}y_t \mid \mathcal{I}_{t+\mathrm{d}t}^q)} = \frac{\operatorname{Cov}(v_1, \beta_t(v_t - p_t) \, \mathrm{d}t + \, \mathrm{d}u_t \mid \mathcal{I}_{t+\mathrm{d}t}^q)}{\operatorname{Var}(\beta_t(v_t - p_t) \, \mathrm{d}t + \, \mathrm{d}u_t \mid \mathcal{I}_{t+\mathrm{d}t}^q)} = \frac{\beta_t \Sigma_t}{\sigma_u^2}, \quad (49)$$

where $\Sigma_t = \mathsf{E}((v_t - p_t)^2 | \mathcal{I}_t^p)$.²⁶ The information set of the informed trader, \mathcal{J}_t^p , is a refinement of the market maker's information set, \mathcal{I}_t^p . Therefore, by the law of iterated expectations, Σ_t satisfies the same equation as V_t :

$$\Sigma_t' = -2\lambda\beta_t \Sigma_t + (1-\mu)^2 \sigma_v^2 + \mu^2 \sigma_e^2 + \lambda^2 \sigma_u^2,$$
(50)

except that it has a different initial condition. If we solve this first order differential equation explicitly, it follows that the transversality condition $V_1 = 0$ is equivalent to $\int_0^1 \beta_t \, dt = +\infty$, and in turn this is equivalent to $\Sigma_1 = 0$. By (49), we get $\beta_t \Sigma_t = \lambda \sigma_u^2$ is constant. Equation (50) then implies that Σ'_t is constant. From $\Sigma_1 = 0$, we get $\Sigma_t =$

²⁵The condition $\lambda_{\tau} > 0$ is also a second order condition with respect to the choice of β_{τ} . To see this, suppose $\lambda_{\tau} < 0$. Then if $\beta_{\tau} > 0$ is chosen very large, equation (44) shows that V_{τ} is very large as well, and thus $\beta_{\tau}V_{\tau}$ can be made arbitrarily large. Thus, there would be no maximum.

²⁶Because $\mathcal{I}_{t+dt}^q = \mathcal{I}_t^p \cup \{dz_t\}$, the two information sets differ only by the infinitesimal quantity dz_t , and thus we can also write $\Sigma_t = \mathsf{E}((v_t - p_t)^2 | \mathcal{I}_{t+dt}^q) = \mathsf{E}((v_t - p_t)^2 | \mathcal{I}_t^p)$.

 $(1-t)\Sigma_0$, and $\beta_t = \frac{\beta_0}{1-t}$. Then, (50) becomes $-\Sigma_0 = -2\lambda^2 \sigma_u^2 + (1-\mu)^2 \sigma_v^2 + \mu^2 \sigma_e^2 + \lambda^2 \sigma_u^2$. Since $\mu = \frac{\sigma_v^2}{\sigma_v^2 + \sigma_e^2}$, we get $\lambda^2 \sigma_u^2 = \Sigma_0 + \frac{\sigma_v^2 \sigma_e^2}{\sigma_v^2 + \sigma_e^2}$, which implies (15). Then, $\beta_0 = \frac{\lambda \sigma_u^2}{\Sigma_0}$ and $\beta_t = \frac{\beta_0}{1-t}$ imply (13).

Fast model: The informed trader has the same objective function as in (40):

$$\pi_t = \mathsf{E}\left(\int_t^1 (v_1 - p_{\tau + d\tau}) \, \mathrm{d}x_\tau \mid \mathcal{J}_t^p\right).$$
(51)

but here we use q_t instead of p_t as a state variable. From (18), we obtain

$$q_{\tau+d\tau} = \mu_{\tau}^{F} dz_{\tau} + m_{\tau}^{F} dy_{\tau}, \quad \text{with}$$

$$m_{\tau}^{F} = \lambda_{\tau}^{F} - \mu_{\tau}^{F} \rho_{\tau}^{F}.$$
(52)

As explained in the discussion before Theorem 1, we consider only strategies dx_{τ} of the type (17), $dx_{\tau} = \beta_{\tau}^{F}(v_{\tau} - q_{\tau}) d\tau + \gamma_{\tau}^{F} dv_{\tau}$. For $\tau \ge t$, denote by

$$V_{\tau} = \mathsf{E}\big((v_{\tau} - q_{\tau})^2 \mid \mathcal{J}_t^p\big).$$
(53)

For convenience, we now omit the superscript F for the coefficients $\beta, \gamma, \mu, \lambda, \rho, m$. The expected profit is

$$\pi_t = \left(\int_t^1 (v_\tau + dv_\tau - q_\tau - \lambda_\tau dx_\tau) dx_\tau \mid \mathcal{J}_t^p \right) = \int_t^1 \left(\beta_\tau V_\tau + (1 - \lambda_\tau \gamma_\tau) \gamma_\tau \sigma_v^2 \right) d\tau.$$
(54)

 V_{τ} is computed as in the benchmark model, except that λ_{τ} is replaced by m_{τ} :

$$V_{\tau + d\tau} = \mathsf{E} \left((v_{\tau + d\tau} - q_{\tau + d\tau})^2 \mid \mathcal{J}_t^p \right) = V_{\tau} + (1 - \mu_{\tau} - m_{\tau} \gamma_{\tau})^2 \sigma_v^2 \, \mathrm{d}\tau + \mu_{\tau}^2 \sigma_e^2 \, \mathrm{d}\tau + m_{\tau}^2 \sigma_u^2 \, \mathrm{d}\tau - 2m_t \beta_t V_{\tau} \, \mathrm{d}\tau.$$
(55)

therefore the law of motion of V_{τ} is a first order differential equation

$$V_{\tau}' = -2m_t \beta_t V_{\tau} + (1 - \mu_{\tau} - m_{\tau} \gamma_{\tau})^2 \sigma_v^2 + \mu_{\tau}^2 \sigma_e^2 + m_{\tau}^2 \sigma_u^2,$$
(56)

or equivalently $\beta_{\tau}V_{\tau} = \frac{-V'_{\tau} + (1 - \mu_{\tau} - m_{\tau}\gamma_{\tau})^2 \sigma_v^2 + \mu_{\tau}^2 \sigma_e^2 + m_{\tau}^2 \sigma_u^2}{2m_{\tau}}$. Substitute this into (40) and integrate by parts. Since $V_t = 0$, we get

$$\pi_{t} = -\frac{V_{1}}{2m_{1}} + \int_{t}^{1} V_{\tau} \left(\frac{1}{2m_{\tau}}\right)' d\tau + \int_{t}^{1} \left(\frac{(1-\mu_{\tau}-m_{\tau}\gamma_{\tau})^{2}\sigma_{v}^{2} + \mu_{\tau}^{2}\sigma_{e}^{2} + m_{\tau}^{2}\sigma_{u}^{2}}{2m_{\tau}} + (1-\lambda_{\tau}\gamma_{\tau})\gamma_{\tau}\sigma_{v}^{2}\right) d\tau.$$
(57)

Since $V_{\tau} > 0$ can be arbitrarily chosen, in order to get an optimum we must have $\left(\frac{1}{2m_{\tau}}\right)' = 0$, which is equivalent to $m_{\tau} = \text{constant}$. For a maximum, the transversality condition $V_1 = 0$ must be also satisfied.

We next turn to the choice of γ_{τ} . The first order condition is

$$-(1-\mu_{\tau}-m_{\tau}\gamma_{\tau})+(1-\lambda_{\tau}\gamma_{\tau})-\lambda_{\tau}\gamma_{\tau} = 0 \implies \gamma_{\tau} = \frac{\mu_{\tau}}{2\lambda_{\tau}-m_{\tau}} = \frac{\mu_{\tau}}{\lambda_{\tau}+\mu_{\tau}\rho_{\tau}}.$$
(58)

Thus, we obtain a nonzero news trading component. The second order condition is $\lambda_{\tau} + \mu_{\tau}\rho_{\tau} > 0$. There is also a second order condition with respect to β : $m_{\tau} > 0$: see Footnote 25.

Next, we derive the pricing rules from the market maker's zero profit conditions. As in the benchmark model, we compute

$$\lambda_{t} = \frac{\operatorname{Cov}_{t}(v_{1}, \mathrm{d}y_{t})}{\operatorname{Var}_{t}(\mathrm{d}y_{t})} = \frac{\operatorname{Cov}_{t}(v_{1}, \beta_{t}(v_{t} - p_{t}) \mathrm{d}t + \gamma_{t} \mathrm{d}v_{t} + \mathrm{d}u_{t})}{\operatorname{Var}(\beta_{t}(v_{t} - p_{t}) \mathrm{d}t + \gamma_{t} \mathrm{d}v_{t} + \mathrm{d}u_{t})} = \frac{\beta_{t}\Sigma_{t} + \gamma_{t}\sigma_{v}^{2}}{\gamma_{t}^{2}\sigma_{v}^{2} + \sigma_{u}^{2}},$$

$$\rho_{t} = \frac{\operatorname{Cov}_{t}(\mathrm{d}z_{t}, \mathrm{d}y_{t})}{\operatorname{Var}_{t}(\mathrm{d}y_{t})} = \frac{\gamma_{t}\sigma_{v}^{2}}{\gamma_{t}^{2}\sigma_{v}^{2} + \sigma_{u}^{2}},$$

$$\mu_{t} = \frac{\operatorname{Cov}_{t}(v_{1}, \mathrm{d}z_{t} - \rho_{t} \mathrm{d}y_{t})}{\operatorname{Var}_{t}(\mathrm{d}z_{t} - \rho_{t} \mathrm{d}y_{t})} = \frac{-\rho_{t}\beta_{t}\Sigma_{t} + (1 - \rho_{t}\gamma_{t})\sigma_{v}^{2}}{(1 - \rho_{t}\gamma_{t})^{2}\sigma_{v}^{2} + \rho_{t}^{2}}\sigma_{u}^{2} + \sigma_{e}^{2}}.$$
(59)

By the same arguments as for the benchmark model, $\Sigma_t = (1-t)\Sigma_0$, $\beta_t = \frac{\beta_0}{1-t}$, and $\beta_t \Sigma_t$, λ_t , ρ_t , μ_t are constant. Since Σ_t satisfies the same equation (56) as V_t , and $\Sigma'_t = -\Sigma_0$, we obtain

$$-\Sigma_0 = -2m_t \beta_t \Sigma_t + (1 - \mu_\tau - m_\tau \gamma_\tau)^2 \sigma_v^2 + \mu_\tau^2 \sigma_e^2 + m_\tau^2 \sigma_u^2.$$
(60)

We now define the following constants:

$$a = \frac{\sigma_u^2}{\sigma_v^2}, \quad b = \frac{\sigma_e^2}{\sigma_v^2}, \quad c = \frac{\Sigma_0}{\sigma_v^2},$$

$$g = \frac{\gamma^2}{a}, \quad \tilde{\lambda} = \lambda\gamma, \quad \tilde{\rho} = \rho\gamma, \quad \nu = \frac{\beta_0 \Sigma_0}{\sigma_u^2}\gamma, \quad \tilde{m} = m\gamma.$$
(61)

With these notations, equations (58)–(60) become

$$\tilde{\lambda} = \mu(1-\tilde{\rho}), \quad \tilde{\lambda} = \frac{\nu+g}{1+g}, \quad \tilde{\rho} = \frac{g}{1+g}, \quad \mu = \frac{1-\nu}{1+b(1+g)}$$

$$c = \frac{2\nu}{g} - (1-\mu-\tilde{m})^2 - \mu^2 b - \frac{\tilde{m}^2}{g}.$$
(62)

Substitute $\tilde{\lambda}$, $\tilde{\rho}$, μ in $\tilde{\lambda} = \mu(1 - \tilde{\rho})$ and solve for ν :

$$\nu = \frac{1 - (1 + b)g - bg^2}{2 + b + bg} = \frac{1 + g}{2 + b + bg} - g.$$
(63)

The other equations, together with $\tilde{m} = \tilde{\lambda} - \mu \tilde{\rho}$, imply

$$\tilde{\lambda} = \frac{1}{2+b+bg}, \quad \tilde{\rho} = \frac{g}{1+g}, \quad \mu = \frac{1+g}{2+b+bg}, \quad \tilde{m} = \frac{1-g}{2+b+bg}, \quad (64)$$

$$1 + c = \frac{(1 + bg)(1 + g)^2}{g(2 + b + bg)^2}.$$
(65)

From (61), we get

$$\gamma = a^{1/2}g^{1/2}, \quad \beta_0 = \frac{\sigma_u^2}{\Sigma_0\gamma}\nu = \frac{a}{c\gamma}\nu = \frac{a^{1/2}}{cg^{1/2}}\nu.$$
 (66)

From (63) and (65), we get $\nu = \frac{1+g}{2+b+bg} - g = \frac{g(2+b+bg)}{(1+g)(1+bg)} \left(\frac{(1+g)^2(1+bg)}{g(2+b+bg)^2} - \frac{(1+g)(1+bg)}{2+b+bg}\right) = \frac{g(2+b+bg)}{(1+g)(1+bg)} \left(c + (1-g)\frac{(1+b+bg)}{2+b+bg}\right)$. We compute $\beta_0 = \frac{a^{1/2}g^{1/2}(2+b+bg)}{(1+g)(1+bg)} \left(1 + \frac{1-g}{c}\frac{1+b+bg}{2+b+bg}\right)$. Using again (65), we get

$$\beta_0 = \frac{a^{1/2}}{(1+c)^{1/2}(1+bg)^{1/2}} \left(1 + \frac{1-g}{c}\frac{1+b+bg}{2+b+bg}\right).$$
(67)

Now substitute a, b, c from (61) in equations (64)–(67) to obtain equations (19)–(24). Moreover, the second order conditions $\lambda + \mu \rho > 0$ and m > 0 are equivalent to $g \in (-1, 1)$.

Finally, we show that the equation $1 + c = \frac{(1+bg)(1+g)^2}{g(2+b+bg)^2}$ has a unique solution $g \in (-1,1)$, which in fact lies in (0,1). This can be shown by noting that

$$F_b(g) = 1 + c$$
, with $F_b(x) = \frac{(1+bx)(1+x)^2}{x(2+b+bx)^2}$. (68)

One verifies $F'_b(x) = \frac{(x+1)(x-1)(2+b+3bx)}{x^2(2+b+bx)^3}$, so $F_b(x)$ decreases on (0,1). Since $F_b(0) = +\infty$

and $F_b(1) = \frac{1}{1+b} < 1$, there is a unique $g \in (0,1)$ so that $F_b(g) = 1 + c.^{27}$

A.2 Useful Comparative Statics

To compare the fast and benchmark models, and to do some comparative statics for the coefficients involved in Theorem 1, we prove the following result.

Lemma 1. With the notations in Theorem 1, the following inequalities are true:

$$\mu^F < \mu^B, \quad \lambda^F > \lambda^B, \quad \beta^F_0 < \beta^B_0. \tag{69}$$

Proof. Recall that in the proof of Theorem 1, we have denoted

$$a = \frac{\sigma_u^2}{\sigma_v^2}, \quad b = \frac{\sigma_e^2}{\sigma_v^2}, \quad c = \frac{\Sigma_0}{\sigma_v^2}.$$
 (70)

We start by showing that

$$\mu^F = \frac{1+g}{2+b+bg} < \mu^B = \frac{1}{1+b}.$$
(71)

By computation, this is equivalent to g < 1, which is true since $g \in (0, 1)$.

We show that

$$\lambda^{F} = \frac{(1+c)^{1/2}}{a^{1/2}} \frac{1}{(1+bg)^{1/2}(1+g)} > \lambda^{B} = \frac{c^{1/2}}{a^{1/2}} \left(1 + \frac{b}{c(b+1)}\right)^{1/2}.$$
 (72)

After squaring the two sides, and using $1 + c = \frac{(1+bg)(1+g)^2}{g(2+b+bg)^2}$, we need to prove that $\frac{1}{g(2+b+bg)^2} > c+1 - \frac{1}{1+b}$, or equivalently $\frac{1}{1+b} > \frac{(1+bg)(1+g)^2}{g(2+b+bg)^2} - \frac{1}{g(2+b+bg)^2} = \frac{2+b+g+2bg+bg^2}{(2+b+bg)^2}$. This reduces to proving 1 + b + (1 - g)(1 + bg) > 0, which is true, since b > 0 and $g \in (0, 1)$.

In the proof of Theorem 1, we have $\nu = \frac{1+g}{2+b+bg} - g = \frac{g(2+b+bg)}{(1+g)(1+bg)} \left(c + (1-g)\frac{(1+b+bg)}{2+b+bg}\right) > 0$. But $\frac{1+g}{2+b+bg} > g$ implies $bg < \frac{1-g}{1+g}$. We now show that

$$\beta_0^F = \frac{a^{1/2}}{cg^{1/2}} \left(\frac{1+g}{2+b+bg} - g \right) < \beta_0^B = \frac{a^{1/2}}{c} \left(c + \frac{b}{1+b} \right)^{1/2}, \tag{73}$$

where we use (63) and (66) for β_0^F , and (13) for β_0^B . Using (65), the desired in-

 $[\]frac{1}{2^{7} \text{One can check that } F_{b}(x) = 1 + c \text{ has no solution on } (-1,0): \text{ When } b \leq 1, F_{b}(x) < 0 \text{ on } (-1,0).$ When $b > 1, F_{b}(x)$ attains its maximum on (-1,0) at $x^{*} = -\frac{2+b}{3b}$, for which $F_{b}(x^{*}) = \frac{(b-1)^{3}}{b(b+2)^{3}} < 1.$

equality is equivalent to $\frac{1}{g} \frac{(1-g-bg-bg^2)^2}{(2+b+bg)^2} < c+1 - \frac{1}{1+b} = \frac{(1+bg)(1+g)^2}{g(2+b+bg)^2} - \frac{1}{1+b}$, or $\frac{1}{1+b} < \frac{4+3b+bg(2-b)-bg^2(1+2b)-b^2g^3}{(2+b+bg)^2}$. After some algebra, this is equivalent to $bg^2(1+g)^2 + bg(1+4g+g^2) < 3+2b$. We use $bg < \frac{1-g}{1+g}$ (proved above) to show that $bg^2(1+g)^2 < g(1-g^2)$ and $bg(1+4g+g^2) < (1-g)\frac{1+4g+g^2}{1+g} < (1-g)(1+3g)$. Then, it is sufficient to prove that $g(1-g^2) + (1-g)(1+3g) < 3+2b$, or $1+3g-3g^2-g^3 < 3+2b$. For this, it is sufficient to prove $1+3g-3g^2 < 3+2b$. But $1+3g-3g^2$ attains its maximum value of $1+\frac{3}{4}$ at $g=\frac{1}{2}$, and $1+\frac{3}{4} < 3+2b$.

A.3 Proof of Proposition 1

See Lemma 1.

A.4 Proof of Proposition 2

For the benchmark model, as in Theorem 1 and Lemma 1, we have

$$\beta_0^B = \frac{\sigma_u}{\Sigma_0^{1/2}} \left(1 + \frac{\sigma_v^2 \sigma_e^2}{\sigma_v^2 + \sigma_e^2} \right)^{1/2} = \frac{a^{1/2}}{c} \left(c + \frac{b}{1+b} \right)^{1/2}.$$
 (74)

From the first equality, since $\frac{\sigma_v^2 \sigma_e^2}{\sigma_v^2 + \sigma_e^2}$ is increasing in σ_v , so is β_0^B . From the second equality, β_0^B is increasing in $a = \frac{\sigma_u^2}{\sigma_v^2}$ and $b = \frac{\sigma_e^2}{\sigma_v^2}$, and decreasing in $c = \frac{\Sigma_0}{\sigma_v^2}$, and thus β_0^B is increasing in σ_u and σ_e , and decreasing in Σ_0 .

As in the proof of Theorem 1, let $F(b, x) = \frac{(1+bx)(1+x)^2}{x(2+b+bx)^2}$, with $\frac{\partial F}{\partial b} = -\frac{(1+x)^2(2+bx+bx^2)}{x(2+b+bx)^3}$ and $\frac{\partial F}{\partial x} = -\frac{(1-x)(1+x)(2+b+3bx)}{x^2(2+b+bx)^3}$. Since $g \in (0,1)$ is the solution of F(b,g(b,c)) = 1+c, by differentiating with respect to b and c, respectively, we get $\frac{\partial F}{\partial b} + \frac{\partial F}{\partial x} \frac{\partial g}{\partial b} = 0$, and $\frac{\partial F}{\partial x} \frac{\partial g}{\partial c} = 1$. We compute

$$\frac{\partial g}{\partial b} = -\frac{g(1+g)(2+bg+bg^2)}{(1-g)(2+b+3bg)}, \quad \frac{\partial g}{\partial c} = -\frac{g^2(2+b+bg)^3}{(1-g)(1+g)(2+b+3bg)}.$$
(75)

This implies that g is decreasing in b and c. Since $\gamma^F = \frac{\sigma_u}{\sigma_v} g^{1/2} = a^{1/2} g^{1/2}$, γ^F is increasing in a (and σ_u); and decreasing in b, c (and σ_e , Σ_0).

From the proof of Theorem 1, we also have

$$\beta_0^F = \frac{a^{1/2}}{cg^{1/2}} \left(\frac{1+g}{2+b+bg} - g \right).$$
(76)

Since g does not depend on a, β_0^F is increasing in a (and σ_u). We use (75) to compute

 $\begin{array}{l} \frac{\partial\beta_0^F}{\partial b} = -\frac{a^{1/2}g^{1/2}(1+g)^2(2+3b+3bg+b^2g+2g^2)}{2c(1-g)(4+4b+8bg+b^2+4b^2g+3b^2g^2)}, \mbox{ thus } \beta_0^F \mbox{ is decreasing in } b \mbox{ (and } \sigma_e). \mbox{ Similarly,} \\ \mbox{we find that } \frac{\partial\beta_0^F}{\partial c} \mbox{ is proportional to } -\frac{1}{c^2} \left(\frac{1+g}{2+b+bg}-g\right) + \frac{g(2+b+bg)(2+b+5bg+b^2g+b^2g^2)}{2c(1-g)(2+b+3bg)}. \mbox{ Substituting } c = \frac{(1+bg)(1+g)^2}{g(2+b+bg)^2} - 1, \mbox{ we obtain that } \frac{\partial\beta_0^F}{\partial c} \mbox{ is proportional to } 2(1-g)(2+b+3bg+b^2g+b^2g^2). \mbox{ Substituting } c = \frac{(1+bg)(1+g)^2}{g(2+b+bg)^2} - 1, \mbox{ we obtain that } \frac{\partial\beta_0^F}{\partial c} \mbox{ is proportional to } 2(1-g)(2+b+3bg+b^2g+b^2g^2). \mbox{ This is a polynomial in } b \mbox{ and } g, \mbox{ which can be written as } -2(1-g)^2 - b^3g(1-g^4) - P(b,g), \mbox{ where } P(b,g) \mbox{ is a polynomial with positive coefficients. Since } b > 0 \mbox{ and } g \in (0,1), \mbox{ we find that } \frac{\partial\beta_0^F}{\partial c} < 0, \mbox{ thus } \beta_0^F \mbox{ is proportional to } b^3g(1+g)^3 + b^2g^3 + 2b^2g^3(1-g) + 13b^2g^2 + 9b^2g + b^2 + 4bg^2 + 4bg + 6bg(1-g^2) + 4g(1-g), \mbox{ which is positive. Thus, } \beta_0^F \mbox{ is increasing in } \sigma_v. \end{array}$

A.5 Proof of Proposition 3

In the benchmark model, Equation (11) and $\gamma_t^B = 0$ imply that $\operatorname{Var}(\mathrm{d}x_t) = (\beta_t^B)^2 \Sigma_t \mathrm{d}t^2 = 0$, since $\mathrm{d}t^2 = 0$. Also, $\operatorname{Var}(\mathrm{d}u_t) = \sigma_u^2 \mathrm{d}t$. Thus, $IPR_t^B = \frac{\operatorname{Var}(\mathrm{d}x_t)}{\operatorname{Var}(\mathrm{d}x_t) + \operatorname{Var}(\mathrm{d}u_t)} = 0$.

In the fast model, Equation (17) implies $\operatorname{Var}(\mathrm{d}x_t) = (\gamma_t^F)^2 \sigma_v^2 \mathrm{d}t$, and Equation (20) implies $(\gamma_t^F)^2 \sigma_v^2 = \sigma_u^2 g$. Therefore, $IPR_t^F = \frac{\sigma_u^2 g \mathrm{d}t}{\sigma_u^2 g \mathrm{d}t + \sigma_u^2 \mathrm{d}t} = \frac{g}{g+1}$. From Theorem 1, we know that $g \in (0, 1)$.

A.6 Proof of Proposition 4

For $k = \{B, F\}$, we write the equilibrium equations

$$dx_t = \beta_t^k (v_t - p_t) dt + \gamma^k dv_t,$$

$$dp_t = \lambda^k dy_t + \mu^k (dz_t - \rho^k dy_t) = m^k dy_t + \mu^k dz_t.$$
(77)

We first prove the following useful result.

Lemma 2. In both the benchmark and the fast models, i.e., if $k \in \{B, F\}$, and for all $s < u \in (0, 1)$,

$$Cov(v_{s} - p_{s}, v_{u} - p_{u}) = \Sigma_{s} \left(\frac{1 - u}{1 - s}\right)^{m^{k}\beta_{0}^{k}},$$

$$\frac{1}{ds}Cov(dv_{s}, v_{u} - p_{u}) = (1 - m^{k}\gamma^{k} - \mu^{k})\sigma_{v}^{2} \left(\frac{1 - u}{1 - s}\right)^{m^{k}\beta_{0}^{k}},$$
(78)

where $m^k = \lambda^k - \mu^k \rho^k$.

Proof. Denote by $X_u = \mathsf{Cov}(v_s - p_s, v_u - p_u)$. For $u \ge s$, $dX_u = \mathsf{Cov}(v_s - p_s, dv_u - dp_u) =$

 $-\operatorname{Cov}(v_s - p_s, \mathrm{d}p_u) = -m^k \beta_u^k X_u \mathrm{d}u = -m^k \frac{\beta_0^k}{1-u} X_u \mathrm{d}u.$ Then, $\operatorname{d}\ln(X_u) = m^k \beta_0^k \mathrm{d}\ln(1-u).$ Also, at u = s, we have $X_s = \Sigma_s$. Thus, we have a first order differential equation, with solution given by the first equation in (78).

Denote by $Y_u = \frac{1}{ds} \operatorname{Cov}(dv_s, v_u - p_u)$. For u > s, $dY_u = \frac{1}{ds} \operatorname{Cov}(dv_s, dv_u - dp_u) = -\frac{1}{ds} \operatorname{Cov}(dv_s, dp_u) = -m^k \beta_u^k Y_u du = -m^k \frac{\beta_0^k}{1-u} Y_u du$. Then, $d\ln(Y_u) = m^k \beta_0^k d\ln(1-u)$. At u = s + ds, we have $Y_{s+ds} = \frac{1}{ds} \operatorname{Cov}(dv_s, v_s - p_s + dv_s - dp_s) = \frac{1}{ds} \operatorname{Cov}(dv_s, dv_s) - \frac{1}{ds} \operatorname{Cov}(dv_s, m^k dy_t + \mu^k dz_t) = (1 - m^k \gamma^k - \mu^k) \sigma_v^2$. Thus, we have a first order differential equation, with solution given by the second equation in (78).

We now prove Proposition 4. For the benchmark model, $m^B = \lambda^B$. Then, using the notations from Lemma 2, we get

$$\operatorname{Corr}(\mathrm{d}x_{t}^{B},\mathrm{d}x_{t+\tau}^{B}) = \frac{\operatorname{Cov}(v_{t}-p_{t},v_{t+\tau}-p_{t+\tau})}{\operatorname{Cov}(v_{t}-p_{t})^{1/2}\operatorname{Cov}(v_{t+\tau}-p_{t+\tau})^{1/2}} = \frac{\sum_{t} \left(\frac{1-t-\tau}{1-t}\right)^{\lambda^{B}\beta_{0}^{B}}}{\sum_{t}^{1/2}\sum_{t+\tau}^{1/2}}.$$
 (79)

Since $\Sigma_s = \Sigma_0(1-s)$, we obtain $\operatorname{Corr}(\operatorname{d} x_t^B, \operatorname{d} x_{t+\tau}^B) = \left(\frac{1-t-\tau}{1-t}\right)^{\lambda^B \beta_0^B - \frac{1}{2}}.^{28}$

In the fast model, we use both equations in (78) to show that the autocovariance of the informed order flow, $Cov(dx_t^F, dx_{t+\tau}^F)$, is of order dt^2 . But the informed order flow variance is of order dt, therefore the autocorrelation is of order dt, which is zero in continuous time.

A.7 Proof of Proposition 5

For $k \in \{B, F\}$, recall that $AT_t^k = \operatorname{Corr}(dx_t, q_{t+dt} - p_{t+dt})$, and $dx_t = \beta_t^k (v_t - p_t) dt + \gamma^k dv_t$. In the benchmark, equation (8) implies $q_{t+dt} - p_{t+dt} = \mu^B dz_{t+dt}$. Therefore, $AT_t^B = 0$.

In the fast model, equation (9) implies $q_{t+dt} - p_{t+dt} = \mu^F (dz_t - \rho^F dy_t)$, thus

$$q_{t+dt} - p_{t+dt} = \mu^{F} (1 - \rho^{F} \gamma^{F}) dv_{t} + \mu^{F} de_{t} - \mu^{F} \rho^{F} du_{t}.$$
 (80)

Then, $\frac{1}{dt} \operatorname{Cov}(dx_t, q_{t+dt} - p_{t+dt}) = \gamma^F \mu^F (1 - \rho^F \gamma^F) \sigma_v^2$. Moreover, $\frac{1}{dt} \operatorname{Var}(dx_t) = (\gamma^F)^2 \sigma_v^2$ and $\frac{1}{dt} \operatorname{Var}(q_{t+dt} - p_{t+dt}) = (\mu^F)^2 (1 - \rho^F \gamma^F)^2 \sigma_v^2 + (\mu^F)^2 \sigma_e^2 + (\mu^F)^2 (\rho^F)^2 \sigma_u^2$. Together, $\frac{1}{2^8 \operatorname{Note that} \lambda^B \beta_0^B = 1 + \frac{\sigma_v^2 \sigma_e^2}{\Sigma_0(\sigma_v^2 + \sigma_e^2)} > 1$. these imply $AT^F = \frac{(1-\rho^F\gamma^F)\sigma_v}{\sqrt{(1-\rho^F\gamma^F)^2\sigma_v^2 + \sigma_e^2 + (\rho^F)^2\sigma_u^2}}$. With $a = \frac{\sigma_u^2}{\sigma_v^2}$ and $b = \frac{\sigma_e^2}{\sigma_v^2}$ as in (61), we use the formulas $\gamma^F = a^{1/2}g^{1/2}$ and $\rho^F = \frac{g^{1/2}}{a^{1/2}(1+g)}$ in Theorem 1 to compute $AT^F = \frac{1}{\sqrt{(1+g)(1+b)}}$. This proves (29) for AT_t^F .

A.8 Proof of Proposition 6

In the fast model, denote by $TV^F = \mathsf{Var}(\mathrm{d}y_t)$ the trading volume, and $IPR^F = \frac{\mathsf{Var}(\mathrm{d}x_t)}{\mathsf{Var}(\mathrm{d}y_t)}$ the informed participation rate. Then, by Proposition 3, $TV^F = \sigma_u^2(1+g)$, and $IPR^F = \frac{g}{1+g}$, hence TV^F and IPR^F have the same dependence on σ_e as g. From (75), g is decreasing in b, hence also in σ_e . Thus, both TV^F and IPR^F are decreasing in σ_e .

As in equation (72), we have $(\lambda^F)^2 = \frac{1+c}{a} \frac{1}{(1+bg)(1+g)^2}$. Using the formula for $\frac{\partial g}{\partial b}$ in (75), we compute $\frac{\partial \left((1+bg)(1+g)^2\right)}{\partial b} = -\frac{g(1+g)^3(1+bg)}{1-g} < 0$. Therefore, λ^F is increasing in *b*, hence in σ_e . Thus, higher precision of the public signal (lower σ_e) implies higher liquidity (lower price impact coefficient λ^F).

A.9 Proof of Proposition 7

As in proof of Proposition 6, the trading volume and the informed participation rate in the fast model are given, respectively, by $TV^F = \sigma_u^2(1+g)$ and $IPR^F = \frac{g}{1+g}$, hence TV^F and IPR^F have the same dependence on σ_v (holding $b = \frac{\sigma_e^2}{\sigma_v^2}$ constant) as g. In the proof of Proposition 2, we analyze g = g(b,c) as a function of $b = \frac{\sigma_e^2}{\sigma_v^2}$ and $c = \frac{\Sigma_0}{\sigma_v^2}$. If we hold b constant, then g depends on σ_v only via c. From equation (75) we have $\frac{\partial g}{\partial c} = -\frac{g^2(2+b+bg)^3}{(1-g)(1+g)(2+b+3bg)} < 0$, thus g is decreasing in c. Therefore, g is increasing in σ_v , hence both TV^F and IPR^F are increasing in σ_v .

Denote by $\Sigma_v = \sigma_v^2$. As in proof of Proposition 6, we use $(\lambda^F)^2 = \frac{1+c}{a} \frac{1}{(1+bg)(1+g)^2} = \frac{\Sigma_v + \Sigma_0}{\sigma_u^2} \frac{1}{(1+bg)(1+g)^2}$. We now hold *b* constant, and differentiate *g* only with respect to $c = \frac{\Sigma_0}{\Sigma_v}$. After substituting also $1 + c = \frac{(1+bg)(1+g)^2}{g(2+b+bg)^2}$, we obtain $\left(\frac{\partial(\lambda^F)^2}{\partial\Sigma_v}\right)_{b=\text{const}} = \frac{1-g+b+bg}{\sigma_u^2(1+g)(1-g)(2+b+bg)} > 0$. Thus, if σ_v increases and $\frac{\sigma_e}{\sigma_v}$ is held constant, λ^F increases.

A.10 Proof of Proposition 9

We discretize the continuous time fast model, and write t + 1 instead of $t + \Delta t$. We also remove the superscript F. Equations (17) and (18) imply

$$\Delta x_t = -\beta_t s_t \Delta t + \gamma \Delta v_t,$$

$$\Delta q_t = m \Delta y_t + \mu \Delta z_t,$$
(81)

where $s_t = q_t - v_t$ is the pricing error, and $m = \lambda - \mu \rho$. Then, using (81), we rewrite $s_{t+1} = s_t + \Delta q_t - \Delta v_t$ as follows:

$$s_{t+1} = (1 - m\beta_t \Delta t)s_t - (1 - \mu - m\gamma)\Delta v_t + (m\Delta u_t + \mu\Delta e_t).$$
(82)

Using equations (64), we can also compute

$$\gamma = a^{1/2}g^{1/2}, \quad \lambda = \frac{1}{\gamma}\frac{1}{2+b+bg}, \quad \rho = \frac{1}{\gamma}\frac{g}{1+g}, \quad \mu = \frac{1+g}{2+b+bg},$$

$$m = \frac{1}{\gamma}\frac{1-g}{2+b+bg}, \quad 1-\mu-m\gamma = \frac{b+bg}{2+b+bg}.$$
(83)

A.11 Proof of Proposition 10

See Lemma 1.

A.12 Proof of Proposition 11

As in Proposition 9, we discretize the models, and write t + 1 instead of $t + \Delta t$. Denote by $\Delta p_t = p_{t+1} - p_t$ the price change over one interval. Equation (8) implies that in the benchmark model $\Delta p_t = \mu^B \Delta z_t + \lambda^B \Delta y_t$; moreover, the benchmark informed order flow is of order Δt , i.e., $\Delta x_t = O(\Delta t)$. Equation (9) implies that in the fast model, $\Delta p_t = \lambda^F \Delta y_t + \mu^F (\Delta z_{t-1} - \rho^F \Delta y_{t-1})$. We aggregate price changes over N intervals from t to t + N, and we denote the sum by Δ_N . We assume that N is large, with $N = O(\frac{1}{\Delta t^{1/3}})$. Then, we obtain, respectively,

$$\Delta_{N}p_{t} = \mu^{B}\Delta_{N}z_{t} + \lambda^{B}\Delta_{N}u_{t} + O(N\Delta t),$$

$$\Delta_{N}p_{t} = \lambda^{F}\Delta_{N}y_{t} + \mu^{F}(\Delta_{N}z_{t-1} - \rho^{F}\Delta_{N}y_{t-1})$$

$$= \lambda^{F}\Delta_{N}y_{t} + \mu^{F}(\Delta_{N}z_{t} - \rho^{F}\Delta_{N}y_{t}) + O(\Delta t^{1/2}),$$
(84)

since $z_t - z_{t-1}$ and $y_t - y_{t-1}$ are of order $\Delta t^{1/2}$. Denote by $\Delta_N t = N\Delta t = O(\Delta t^{2/3})$. Note that $\Delta_N u_t = O(N^{1/2}\Delta t^{1/2}) = O(\Delta t^{1/3}) = O((\Delta_N t)^{1/2})$, and similarly for the other independent diffusion processes. We have $O(N\Delta t) = O(\Delta_N t)$ and $O(\Delta t^{1/2}) = o(\Delta_N t)$. For the benchmark model we obtain

$$\frac{1}{\sigma_v^2 \Delta_N t} \operatorname{Cov}(\Delta_N p_t, \Delta_N v_t) = \mu^B + \frac{o(\Delta_N t)}{\Delta_N t}.$$
(85)

In the fast model, the informed order flow is $\Delta x_t = \gamma^F \Delta v_t + O(\Delta t)$, and we compute

$$\frac{1}{\sigma_v^2 \Delta_N t} \operatorname{Cov}(\Delta_N p_t, \Delta_N v_t) = \gamma^F (\lambda^F - \mu^F \rho^F) + \mu^B + \frac{o(\Delta_N t)}{\Delta_N t}.$$
(86)

Finally, we prove that $\gamma^F(\lambda^F - \mu^F \rho^F) + \mu^B > \mu^B$. Using (83) and (71), we need to show that $\frac{2}{2+b+bg} > \frac{1}{1+b}$, which is equivalent to 1 > g. But this is true, since $g \in (0, 1)$.

A.13 Proof of Proposition 12

Denote $\operatorname{Var}(\mathrm{d}p_t) = \sigma_p^2 \mathrm{d}t$ the variance of the instantaneous price changes, and we use Theorem 1 to compute the various components of this price change. In the benchmark model, $\operatorname{Var}(p_{t+\mathrm{d}t} - q_t) = (\lambda^B)^2 \sigma_u^2 \mathrm{d}t = \left(\Sigma_0 + \frac{\sigma_v^2 \sigma_e^2}{\sigma_v^2 + \sigma_e^2}\right) \mathrm{d}t$. Also, $\operatorname{Var}(q_{t+\mathrm{d}t} - p_{t+\mathrm{d}t}) = (\mu^B)^2 (\sigma_v^2 + \sigma_e^2) \mathrm{d}t = \frac{\sigma_v^4}{\sigma_v^2 + \sigma_e^2} \mathrm{d}t$. We obtain the volatility decomposition in the benchmark model,

$$\sigma_p^2 = \Sigma_0 + \sigma_v^2 = \left(\Sigma_0 + \frac{\sigma_v^2 \sigma_e^2}{\sigma_v^2 + \sigma_e^2}\right) + \frac{\sigma_v^4}{\sigma_v^2 + \sigma_e^2}.$$
(87)

Similarly, in the fast model, $\operatorname{Var}(p_{t+dt} - q_t) = (\lambda^F)^2 ((\gamma^F)^2 \sigma_v^2 + \sigma_u^2) dt$. Using equation (83), we compute $\operatorname{Var}(p_{t+dt} - q_t) = \frac{1+g}{g(2+b+bg)^2} \sigma_v^2 dt$. Also, $\operatorname{Var}(q_{t+dt} - p_{t+dt}) = (\mu^F)^2 ((1 - \rho^F \gamma^F)^2 \sigma_v^2 + \sigma_e^2 + (\rho^F)^2 \sigma_u^2) dt = \frac{(1+g)(1+b+bg)}{(2+b+bg)^2} \sigma_v^2 dt$.

Using the equilibrium parameter values of Theorem 1, we obtain that $\operatorname{Var}(p_{t+dt} - q_t)$ is higher than in the benchmark, $\operatorname{Var}(q_{t+dt} - p_{t+dt})$ is lower than in the benchmark, and $\operatorname{Var}(dp_t) = \Sigma_0 + \sigma_v^2 t$ is the same as in the benchmark. We obtain the volatility decomposition in the fast model,

$$\sigma_p^2 = \Sigma_0 + \sigma_v^2 = \frac{1+g}{g(2+b+bg)^2} \sigma_v^2 + \frac{(1+g)(1+b+bg)}{(2+b+bg)^2} \sigma_v^2.$$
(88)

we note that, according to (65), $\Sigma_0 + \sigma_v^2 = \sigma_v^2 (1+c) = \sigma_v^2 \frac{(1+g)^2 (1+bg)}{g(2+b+bg)^2}$.

Finally, we show that the volatility component coming from quote updates is larger

in the benchmark, i.e., $\frac{\sigma_v^2}{\sigma_v^2 + \sigma_e^2} = \frac{1}{1+b} > \frac{(1+g)(1+b+bg)}{(2+b+bg)^2}$. The difference is proportional to $3 - g + 2b + bg - bg^2 = 2(1+b) + (1-g)(1+bg) > 0$. Since the total volatility is the same, it also implies that the volatility component coming from the trades is larger in the fast model.

B Models in Discrete Time

B.1 Discrete Time Fast Model

We divide the interval [0, 1] into T equally spaced intervals of length $\Delta t = \frac{1}{T}$. Trading takes place at equally spaced times, $t = 1, 2, \ldots, T - 1$. The sequence of events is as follows. At t = 0, the informed trader observes v_0 . At each $t = 1, \ldots, T - 1$, the informed trader observes $\Delta v_t = v_t - v_{t-1}$; and the market maker observes $\Delta z_{t-1} =$ $\Delta v_{t-1} + \Delta e_{t-1}$, except at t = 1. The error in the market maker's signal is normally distributed, $\Delta e_{t-1} \sim \mathcal{N}(0, \sigma_e^2 \Delta t)$. The market maker quotes the bid price = the ask price = q_t . The informed trader then submits Δx_t , and the liquidity traders submit in aggregate $\Delta u_t \sim \mathcal{N}(0, \sigma_u^2 \Delta t)$. The market maker observes only the aggregate order flow, $\Delta y_t = \Delta x_t + \Delta u_t$, and sets the price at which the trading takes place, p_t . The market maker is competitive, i.e., makes zero profit. This translates into the following formulas:

$$p_{t} = \mathsf{E}(v_{t} \mid \mathcal{I}_{t}^{p}), \qquad \mathcal{I}_{t}^{p} = \{\Delta y_{1}, \dots, \Delta y_{t}, \Delta z_{1}, \dots, \Delta z_{t-1}\},$$

$$q_{t+1} = \mathsf{E}(v_{t} \mid \mathcal{I}_{t}^{q}), \qquad \mathcal{I}_{t}^{q} = \{\Delta y_{1}, \dots, \Delta y_{t}, \Delta z_{1}, \dots, \Delta z_{t}\}.$$
(89)

We also denote

$$\Omega_t = \operatorname{Var}(v_t \mid \mathcal{I}_t^p),$$

$$\Sigma_t = \operatorname{Var}(v_t \mid \mathcal{I}_t^q).$$
(90)

Definition 1. A pricing rule p_t is called linear if it is of the form $p_t = q_t + \lambda_t \Delta y_t$, for all $t = 1, \ldots, T - 1$.²⁹ An equilibrium is called linear if the pricing rule is linear, and the informed trader's strategy Δx_t is linear in $\{v_\tau\}_{\tau \leq t}$ and $\{q_\tau\}_{\tau \leq t}$.

The next result shows that if the pricing rule is linear, the informed trader's strategy

²⁹We could defined more generally, a pricing rule to be linear in the whole history $\{\Delta y_{\tau}\}_{\tau \leq t}$, but as Kyle (1985) shows, this is equivalent to the pricing rule being linear only in Δy_t .

is also linear, and furthermore it can be decomposed into a forecast error component, $\beta_t(v_{t-1} - q_t)$, and a news trading component, $\gamma_t \Delta v_t$.

Theorem 2. Any linear equilibrium must be of the form

$$\Delta x_t = \beta_t (v_{t-1} - q_t) \Delta t + \gamma_t \Delta v_t,$$

$$p_t = q_t + \lambda_t \Delta y_t,$$

$$q_{t+1} = p_t + \mu_t (\Delta z_t - \rho_t \Delta y_t),$$
(91)

for t = 1, ..., T - 1, where β_t , γ_t , λ_t , μ_t , ρ_t , Ω_t , and Σ_t are constants that satisfy

$$\lambda_{t} = \frac{\beta_{t}\Sigma_{t-1} + \gamma_{t}\sigma_{v}^{2}}{\beta_{t}^{2}\Sigma_{t-1}\Delta t + \gamma_{t}^{2}\sigma_{v}^{2} + \sigma_{u}^{2}},$$

$$\mu_{t} = \frac{(\sigma_{u}^{2} + \beta_{t}^{2}\Sigma_{t-1}\Delta t - \beta_{t}\gamma_{t}\Sigma_{t-1})\sigma_{v}^{2}}{(\beta_{t}^{2}\Sigma_{t-1}\Delta t + \gamma_{t}^{2}\sigma_{v}^{2} + \sigma_{u}^{2})\sigma_{e}^{2} + (\beta_{t}^{2}\Sigma_{t-1}\Delta t + \sigma_{u}^{2})\sigma_{v}^{2}},$$

$$m_{t} = \lambda_{t} - \rho_{t}\mu_{t} = \frac{\beta_{t}\Sigma_{t-1}(\sigma_{v}^{2} + \sigma_{e}^{2}) + \gamma_{t}\sigma_{v}^{2}\sigma_{e}^{2}}{(\beta_{t}^{2}\Sigma_{t-1}\Delta t + \gamma_{t}^{2}\sigma_{v}^{2} + \sigma_{u}^{2})\sigma_{e}^{2} + (\beta_{t}^{2}\Sigma_{t-1}\Delta t + \sigma_{u}^{2})\sigma_{v}^{2}},$$

$$\rho_{t} = \frac{\gamma_{t}\sigma_{v}^{2}}{\beta_{t}^{2}\Sigma_{t-1}\Delta t + \gamma_{t}^{2}\sigma_{v}^{2} + \sigma_{u}^{2}},$$

$$\Omega_{t} = \Sigma_{t-1} + \sigma_{v}^{2}\Delta t - \frac{\beta_{t}^{2}\Sigma_{t-1}^{2} + 2\beta_{t}\gamma_{t}\Sigma_{t-1}\sigma_{v}^{2} + \gamma_{t}^{2}\sigma_{v}^{4}}{\beta_{t}^{2}\Sigma_{t-1}\Delta t + \gamma_{t}^{2}\sigma_{v}^{2} + \sigma_{u}^{2}}\Delta t,$$

$$\Sigma_{t} = \Sigma_{t-1} + \sigma_{v}^{2}\Delta t - \frac{\beta_{t}^{2}\Sigma_{t-1}^{2} - (\Delta t \sigma_{v}^{4} + \sigma_{v}^{4}\sigma_{u}^{2} + \gamma_{t}^{2}\sigma_{v}^{4}\sigma_{e}^{2} + 2\beta_{t}\gamma_{t}\Sigma_{t-1}\sigma_{v}^{2}\sigma_{e}^{2}}{(\beta_{t}^{2}\Sigma_{t-1}\Delta t + \gamma_{t}^{2}\sigma_{v}^{2} + \sigma_{u}^{2})\sigma_{e}^{2} + (\beta_{t}^{2}\Sigma_{t-1}\Delta t + \sigma_{u}^{2})\sigma_{v}^{2}}\Delta t.$$

$$(92)$$

The value function of the informed trader is quadratic for all t = 1, ..., T - 1:

$$\pi_t = \alpha_{t-1}(v_{t-1} - q_t)^2 + \alpha'_{t-1}(\Delta v_t)^2 + \alpha''_{t-1}(v_{t-1} - q_t)\Delta v_t + \delta_{t-1}.$$
(93)

The coefficients of the optimal trading strategy and the value function satisfy

$$\beta_t \Delta t = \frac{1 - 2\alpha_t m_t}{2(\lambda_t - \alpha_t m_t^2)},$$

$$\gamma_t = \frac{1 - 2\alpha_t m_t (1 - \mu_t)}{2(\lambda_t - \alpha_t m_t^2)},$$

$$\alpha_{t-1} = \beta_t \Delta t (1 - \lambda_t \beta_t \Delta t) + \alpha_t (1 - m_t \beta_t \Delta t)^2,$$

$$\alpha'_{t-1} = \alpha_t (1 - \mu_t - m_t \gamma_t)^2 + \gamma_t (1 - \lambda_t \gamma_t),$$

$$\alpha''_{t-1} = \beta_t \Delta t + \gamma_t (1 - 2\lambda_t \beta_t \Delta t) + 2\alpha_t (1 - m_t \beta_t \Delta t) (1 - \mu_t - m_t \gamma_t),$$

$$\delta_{t-1} = \alpha_t (m_t^2 \sigma_u^2 + \mu_t^2 \sigma_e^2) \Delta t + \alpha'_t \sigma_v^2 \Delta t + \delta_t.$$

(94)

The terminal conditions are

$$\alpha_T = \alpha'_T = \alpha''_T = \delta_T = 0. \tag{95}$$

The second order condition is

$$\lambda_t - \alpha_t m_t^2 > 0. (96)$$

Given Σ_0 , conditions (92)–(96) are necessary and sufficient for the existence of a linear equilibrium.

Proof. First, we show that equations (92) are equivalent to the zero profit conditions of the market maker. Second, we show that equations (94)–(96) are equivalent to the informed trader's strategy in (91) being optimal.

Zero Profit of market maker: Let us start with with the market maker's update due to the order flow at t = 1, ..., T - 1. Conditional on \mathcal{I}_{t-1}^q , the variables $v_{t-1} - q_t$ and Δv_t have a bivariate normal distribution:

$$\begin{bmatrix} v_{t-1} - q_t \\ \Delta v_t \end{bmatrix} | \mathcal{I}_{t-1}^q \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_{t-1} & 0 \\ 0 & \sigma_v^2 \end{bmatrix} \right).$$
(97)

The aggregate order flow at t is of the form

$$\Delta y_t = \beta_t (v_{t-1} - q_t) \Delta t + \gamma_t \Delta v_t + \Delta u_t.$$
(98)

Denote by

$$\Phi_t = \mathsf{Cov}\left(\left[\begin{array}{c} v_{t-1} - q_t \\ \Delta v_t \end{array}\right], \Delta y_t\right) = \left[\begin{array}{c} \beta_t \Sigma_{t-1} \\ \gamma_t \sigma_v^2 \end{array}\right] \Delta t.$$
(99)

Then, conditional on $\mathcal{I}_t = \mathcal{I}_{t-1}^q \cup \{\Delta y_t\}$, the distribution of $v_{t-1} - q_t$ and Δv_t is bivariate normal:

$$\begin{bmatrix} v_{t-1} - q_t \\ \Delta v_t \end{bmatrix} | \mathcal{I}_t \sim \mathcal{N}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \right), \quad (100)$$

where

$$\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \Phi_t \operatorname{Var}(\Delta y_t)^{-1} \Delta y_t = \begin{bmatrix} \beta_t \Sigma_{t-1} \\ \gamma_t \sigma_v^2 \end{bmatrix} \frac{1}{\beta_t^2 \Sigma_{t-1} \Delta t + \gamma_t^2 \sigma_v^2 + \sigma_u^2} \Delta y_t, \quad (101)$$

and

$$\begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix} = \operatorname{Var} \left(\begin{bmatrix} v_{t-1} - q_t \\ \Delta v_t \end{bmatrix} \right) - \Phi_t \operatorname{Var} (\Delta y_t)^{-1} \Phi_t'$$

$$= \begin{bmatrix} \Sigma_{t-1} & 0 \\ 0 & \sigma_v^2 \Delta t \end{bmatrix} - \frac{1}{\beta_t^2 \Sigma_{t-1} \Delta t + \gamma_t^2 \sigma_v^2 + \sigma_u^2} \begin{bmatrix} \beta_t^2 \Sigma_{t-1}^2 & \beta_t \gamma_t \Sigma_{t-1} \sigma_v^2 \\ \beta_t \gamma_t \Sigma_{t-1} \sigma_v^2 & \gamma_t^2 \sigma_v^4 \end{bmatrix} \Delta t.$$
(102)

We compute

$$p_t - q_t = \mathsf{E}(v_t - q_t \mid \mathcal{I}_t) = \mu_1 + \mu_2 = \frac{\beta_t \Sigma_{t-1} + \gamma_t \sigma_v^2}{\beta_t^2 \Sigma_{t-1} \Delta t + \gamma_t^2 \sigma_v^2 + \sigma_u^2} \,\Delta y_t, \tag{103}$$

which proves equation (92) for λ_t . Also,

$$\Omega_t = \operatorname{Var}(v_t - q_t \mid \mathcal{I}_t) = \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2$$

= $\Sigma_{t-1} + \sigma_v^2 \Delta t - \frac{\beta_t^2 \Sigma_{t-1}^2 + 2\beta_t \gamma_t \Sigma_{t-1} \sigma_v^2 + \gamma_t^2 \sigma_v^4}{\beta_t^2 \Sigma_{t-1} \Delta t + \gamma_t^2 \sigma_v^2 + \sigma_u^2} \Delta t,$ (104)

which proves the formula for Ω_t .

Next, to compute $q_{t+1} = \mathsf{E}(v_t \mid \mathcal{I}_t^q)$, we start from the same prior as in (97), but we consider the impact of both the order flow at t and the market maker's signal at t + 1:

$$\Delta y_t = \beta_t (v_{t-1} - q_t) \Delta t + \gamma_t \Delta v_t + \Delta u_t,$$

$$\Delta z_t = \Delta v_t + \Delta e_t.$$
(105)

Denote by

$$\Psi_{t} = \operatorname{Cov}\left(\left[\begin{array}{c} v_{t-1} - q_{t} \\ \Delta v_{t} \end{array}\right], \left[\begin{array}{c} \Delta y_{t} \\ \Delta z_{t} \end{array}\right]\right) = \left[\begin{array}{c} \beta_{t}\Sigma_{t-1} & 0 \\ \gamma_{t}\sigma_{v}^{2} & \sigma_{v}^{2} \end{array}\right]\Delta t,$$

$$V_{t}^{yz} = \operatorname{Var}\left(\left[\begin{array}{c} \Delta y_{t} \\ \Delta z_{t} \end{array}\right]\right) = \left[\begin{array}{c} \beta_{t}^{2}\Sigma_{t-1}\Delta t + \gamma_{t}^{2}\sigma_{v}^{2} + \sigma_{u}^{2} & \gamma_{t}\sigma_{v}^{2} \\ \gamma_{t}\sigma_{v}^{2} & \sigma_{v}^{2} + \sigma_{e}^{2} \end{array}\right]\Delta t.$$
(106)

Conditional on $\mathcal{I}_t^q = \mathcal{I}_{t-1}^q \cup \{\Delta y_t, \Delta z_t\}$, the distribution of $v_{t-1} - q_t$ and Δv_t is bivariate normal:

$$\begin{bmatrix} v_{t-1} - q_t \\ \Delta v_t \end{bmatrix} | \mathcal{I}_t^q \sim \mathcal{N}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}\right), \quad (107)$$

where

$$\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \Psi_t (V_t^{yz})^{-1} \begin{bmatrix} \Delta y_t \\ \Delta z_t \end{bmatrix} = \begin{bmatrix} \beta_t \Sigma_{t-1} (\sigma_v^2 + \sigma_e^2) \Delta y_t - \beta_t \gamma_t \Sigma_{t-1} \sigma_v^2 \Delta z_t \\ \gamma_t \sigma_v^2 \sigma_e^2 \Delta y_t + (\beta_t^2 \Sigma_{t-1} \Delta t + \sigma_u^2) \sigma_v^2 \Delta z_t \end{bmatrix} = \frac{\left[\beta_t \Sigma_{t-1} \Delta t + \gamma_t^2 \sigma_v^2 + \sigma_u^2 \sigma_v^2 + (\beta_t^2 \Sigma_{t-1} \Delta t + \sigma_u^2) \sigma_v^2 + (\beta_t^2 \Sigma_{t-1} \Delta t + \sigma_u^2) \sigma_v^2 \right]}{(\beta_t^2 \Sigma_{t-1} \Delta t + \gamma_t^2 \sigma_v^2 + \sigma_u^2) \sigma_e^2 + (\beta_t^2 \Sigma_{t-1} \Delta t + \sigma_u^2) \sigma_v^2}, (108)$$

and

$$\begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix} = \operatorname{Var} \left(\begin{bmatrix} v_{t-1} - q_t \\ \Delta v_t \end{bmatrix} \right) - \Psi_t \left(V_t^{yz} \right)^{-1} \Psi_t'$$

$$= \begin{bmatrix} \Sigma_{t-1} & 0 \\ 0 & \sigma_v^2 \Delta t \end{bmatrix} - \frac{\begin{bmatrix} \beta_t^2 \Sigma_{t-1}^2 (\sigma_v^2 + \sigma_e^2) & \beta_t \gamma_t \Sigma_{t-1} \sigma_v^2 \sigma_e^2 \\ \beta_t \gamma_t \Sigma_{t-1} \sigma_v^2 \sigma_e^2 & (\beta_t^2 \Sigma_{t-1} \Delta t + \gamma_t^2 \sigma_e^2 + \sigma_u^2) \sigma_v^4 \end{bmatrix} \Delta t.$$
(109)

Therefore,

$$q_{t+1} - q_t = \mu_1 + \mu_2$$

$$= \frac{\left(\beta_t \Sigma_{t-1}(\sigma_v^2 + \sigma_e^2) + \gamma_t \sigma_v^2 \sigma_e^2\right) \Delta y_t + \left(\sigma_u^2 + \beta_t^2 \Sigma_{t-1} \Delta t - \beta_t \gamma_t \Sigma_{t-1}\right) \sigma_v^2 \Delta z_t}{\left(\beta_t^2 \Sigma_{t-1} \Delta t + \gamma_t^2 \sigma_v^2 + \sigma_u^2\right) \sigma_e^2 + \left(\beta_t^2 \Sigma_{t-1} \Delta t + \sigma_u^2\right) \sigma_v^2}$$
(110)

$$= m_t \Delta y_t + \mu_t \Delta z_t = (\lambda_t - \rho_t \mu_t) \Delta y_t + \mu_t \Delta z_t, \qquad (111)$$

which proves equation (92) for μ_t , m_t , and ρ_t . Also,

$$\Sigma_{t} = \sigma_{1}^{2} + \sigma_{2}^{2} + 2\rho\sigma_{1}\sigma_{2}$$

$$= \Sigma_{t-1} + \sigma_{v}^{2}\Delta t - \frac{\beta_{t}^{2}\Sigma_{t-1}^{2}(\sigma_{v}^{2} + \sigma_{e}^{2}) + \beta_{t}^{2}\Sigma_{t-1}\Delta t\sigma_{v}^{4} + \sigma_{v}^{4}\sigma_{u}^{2} + \gamma_{t}^{2}\sigma_{v}^{4}\sigma_{e}^{2} + 2\beta_{t}\gamma_{t}\Sigma_{t-1}\sigma_{v}^{2}\sigma_{e}^{2}}{\left(\beta_{t}^{2}\Sigma_{t-1} + (\beta_{t} + \gamma_{t})^{2}\sigma_{v}^{2} + \sigma_{u}^{2}\right)\sigma_{e}^{2} + \left(\beta_{t}^{2}\Sigma_{t-1} + \sigma_{u}^{2}\right)\sigma_{v}^{2}} \Delta t$$
(112)

which proves the formula for Σ_t .

Optimal Strategy of Informed Trader: At each $t = 1, \ldots, T - 1$, the informed trader maximizes the expected profit: $\pi_t = \max \sum_{\tau=t}^{T-1} \mathsf{E}((v_T - p_{\tau})\Delta x_{\tau})$. We prove by backward induction that the value function is quadratic and of the form given in (93): $\pi_t = \alpha_{t-1}(v_{t-1} - q_t)^2 + \alpha'_{t-1}(\Delta v_t)^2 + \alpha''_{t-1}(v_{t-1} - q_t)\Delta v_t + \delta_{t-1}$. At the last decision point (t = T - 1) the next value function is zero, i.e., $\alpha_T = \alpha'_T = \alpha'_T = \delta_T = 0$, which

are the terminal conditions (95). This is the transversality condition: no money is left on the table. In the induction step, if t = 1, ..., T - 1, we assume that π_{t+1} is of the desired form. The Bellman principle of intertemporal optimization implies

$$\pi_t = \max_{\Delta x} \mathsf{E}\Big((v_t - p_t)\Delta x + \pi_{t+1} \mid \mathcal{I}_t^q, v_t, \Delta v_t\Big).$$
(113)

The last two equations in (91) imply that the quote q_t evolves by $q_{t+1} = q_t + m_t \Delta y_t + \mu_t \Delta z_t$, where $m_t = \lambda_t - \rho_t \mu_t$. This implies that the informed trader's choice of Δx affects the trading price and the next quote by

$$p_t = q_t + \lambda_t (\Delta x + \Delta u_t),$$

$$q_{t+1} = q_t + m_t (\Delta x + \Delta u_t) + \mu_t \Delta z_t.$$
(114)

Substituting these into the Bellman equation, we get

$$\pi_{t} = \max_{\Delta x} \mathsf{E} \Big(\Delta x (v_{t-1} + \Delta v_{t} - q_{t} - \lambda_{t} \Delta x - \lambda_{t} \Delta u_{t}) \\ + \alpha_{t} (v_{t-1} + \Delta v_{t} - q_{t} - m_{t} \Delta x - m_{t} \Delta u_{t} - \mu_{t} \Delta v_{t} - \mu_{t} \Delta e_{t})^{2} + \alpha_{t}' \Delta v_{t+1}^{2} \\ + \alpha_{t}'' (v_{t-1} + \Delta v_{t} - q_{t} - m_{t} \Delta x - m_{t} \Delta u_{t} - \mu_{t} \Delta v_{t} - \mu_{t} \Delta e_{t}) \Delta v_{t+1} + \delta_{t} \Big) \\ = \max_{\Delta x} \Delta x (v_{t-1} - q_{t} + \Delta v_{t} - \lambda_{t} \Delta x) \\ + \alpha_{t} \Big((v_{t-1} - q_{t} - m_{t} \Delta x + (1 - \mu_{t}) \Delta v_{t})^{2} + (m_{t}^{2} \sigma_{u}^{2} + \mu_{t}^{2} \sigma_{e}^{2}) \Delta t \Big) + \alpha_{t}' \sigma_{v}^{2} \Delta t \\ + 0 + \delta_{t}.$$

$$(115)$$

The first order condition with respect to Δx is

$$\Delta x = \frac{1 - 2\alpha_t m_t}{2(\lambda_t - \alpha_t m_t^2)} (v_{t-1} - q_t) + \frac{1 - 2\alpha_t m_t (1 - \mu_t)}{2(\lambda_t - \alpha_t m_t^2)} \Delta v_t,$$
(116)

and the second order condition for a maximum is $\lambda_t - \alpha_t m_t^2 > 0$, which is (96). Thus, the optimal Δx is indeed of the form $\Delta x_t = \beta_t (v_{t-1} - q_t) \Delta t + \gamma_t \Delta v_t$, where $\beta_t \Delta t$ and γ_t are as in (94). We substitute Δx_t in the formula for π_t to obtain

$$\pi_{t} = \left(\beta_{t}\Delta t(1-\lambda_{t}\beta_{t}\Delta t) + \alpha_{t}(1-m_{t}\beta_{t}\Delta t)^{2}\right)(v_{t-1}-q_{t})^{2} + \left(\alpha_{t}(1-\mu_{t}-m_{t}\gamma_{t})^{2} + \gamma_{t}(1-\lambda_{t}\gamma_{t})\right)\Delta v_{t}^{2}$$

$$+ \left(\beta_{t}\Delta t + \gamma_{t}(1-2\lambda_{t}\beta_{t}\Delta t) + 2\alpha_{t}(1-m_{t}\beta_{t}\Delta t)(1-\mu_{t}-m_{t}\gamma_{t})\right)(v_{t-1}-q_{t})\Delta v_{t} + \alpha_{t}\left(m_{t}^{2}\sigma_{u}^{2} + \mu_{t}^{2}\sigma_{e}^{2}\right)\Delta t + \alpha_{t}'\sigma_{v}^{2}\Delta t + \delta_{t}.$$
(117)

This proves that indeed π_t is of the form $\pi_t = \alpha_{t-1}(v_{t-1}-q_t)^2 + \alpha'_{t-1}(\Delta v_t)^2 + \alpha''_{t-1}(v_{t-1}-q_t)\Delta v_t + \delta_{t-1}$, with α_{t-1} , α'_{t-1} , α''_{t-1} and δ_{t-1} as in (94).

We now briefly discuss the existence of a solution for the recursive system given in Theorem 2. The system of equations (92)–(94) can be numerically solved backwards, starting from the boundary conditions (95). We also start with an arbitrary value of $\Sigma_T > 0.^{30}$ By backward induction, suppose α_t and Σ_t are given. One verifies that equation (92) for Σ_t implies

$$\Sigma_{t-1} = \frac{\Sigma_t \left(\sigma_v^2 \sigma_u^2 + \sigma_v^2 (\sigma_u^2 + \gamma_t^2 \sigma_e^2)\right) - \sigma_v^2 \sigma_u^2 \sigma_e^2 \Delta t}{\left(\sigma_u^2 \sigma_e^2 + \sigma_v^2 (\sigma_u^2 + \gamma_t^2 \sigma_e^2) + \beta_t^2 \Delta t^2 \sigma_v^2 \sigma_e^2 - 2\gamma_t \beta_t \Delta t \sigma_v^2 \sigma_e^2\right) - \Sigma_t \beta_t^2 \Delta t \left(\sigma_v^2 + \sigma_e^2\right)}.$$
(118)

Then, in equation (92) we can rewrite λ_t, μ_t, m_t as functions of $(\Sigma_t, \beta_t, \gamma_t)$ instead of $(\Sigma_{t-1}, \beta_t, \gamma_t)$. Next, we use the formulas for β_t and γ_t to express λ_t, μ_t, m_t as functions of $(\lambda_t, \mu_t, m_t, \alpha_t, \Sigma_t)$. This gives a system of polynomial equations, whose solution λ_t, μ_t, m_t depends only on (α_t, Σ_t) . Numerical simulations show that the solution is unique under the second order condition (96), but the authors have not been able to prove theoretically that this is true in all cases. Once the recursive system is computed for all $t = 1, \ldots, T - 1$, the only condition left to do is to verify that the value obtained for Σ_0 is the correct one. However, unlike in Kyle (1985), the recursive equation for Σ_t is not linear, and therefore the parameters cannot be simply rescaled. Instead, one must numerically modify the initial choice of Σ_T until the correct value of Σ_0 is reached.

B.2 Discrete Time Benchmark Model

The setup is the same as for the fast model, except that the market maker gets the signal Δz at the same time as the informed trader observes Δv . The sequence of events

³⁰Numerically, it should be of the order of Δt .

is as follows. At t = 0, the informed trader observes v_0 . At each $t = 1, \ldots, T - 1$, the informed trader observes $\Delta v_t = v_t - v_{t-1}$; and the market maker observes $\Delta z_t = \Delta v_t + \Delta e_t$, with $\Delta e_t \sim \mathcal{N}(0, \sigma_e^2 \Delta t)$. The market maker quotes the bid price = the ask price = q_t . The informed trader then submits Δx_t , and the liquidity traders submit in aggregate $\Delta u_t \sim \mathcal{N}(0, \sigma_u^2 \Delta t)$. The market maker observes only the aggregate order flow, $\Delta y_t = \Delta x_t + \Delta u_t$, and sets the price at which the trading takes place, p_t . The market maker is competitive, i.e., makes zero profit. This implies

$$p_t = \mathsf{E}(v_t \mid \mathcal{I}_t^p), \qquad \mathcal{I}_t^p = \{\Delta y_1, \dots, \Delta y_t, \Delta z_1, \dots, \Delta z_t\},$$

$$q_t = \mathsf{E}(v_t \mid \mathcal{I}_t^q), \qquad \mathcal{I}_t^q = \{\Delta y_1, \dots, \Delta y_{t-1}, \Delta z_1, \dots, \Delta z_t\}.$$
(119)

We also denote

$$\Sigma_t = \operatorname{Var}(v_t \mid \mathcal{I}_t^p),$$

$$\Omega_t = \operatorname{Var}(v_t \mid \mathcal{I}_t^q).$$
(120)

The next result shows that if the pricing rule is linear, the informed trader's strategy is also linear, and furthermore it only has a forecast error component, $\beta_t(v_t - q_t)$.

Theorem 3. Any linear equilibrium must be of the form

$$\Delta x_t = \beta_t (v_t - q_t) \Delta t,$$

$$p_t = q_t + \lambda_t \Delta y_t,$$

$$q_t = p_{t-1} + \mu_{t-1} \Delta z_t,$$

(121)

for t = 1, ..., T - 1, where by convention $p_0 = 0$, and β_t , γ_t , λ_t , μ_t , Ω_t , and Σ_t are constants that satisfy

$$\lambda_{t} = \frac{\beta_{t}\Sigma_{t}}{\sigma_{u}^{2}},$$

$$\mu_{t} = \frac{\sigma_{v}^{2}}{\sigma_{v}^{2} + \sigma_{e}^{2}},$$

$$\Omega_{t} = \frac{\Sigma_{t}\sigma_{u}^{2}}{\sigma_{u}^{2} - \beta_{t}^{2}\Sigma_{t}\Delta t},$$

$$\Sigma_{t-1} = \Sigma_{t} + \frac{\beta_{t}^{2}\Sigma_{t}^{2}}{\sigma_{u}^{2} - \beta_{t}^{2}\Sigma_{t}\Delta t} \Delta t - \frac{\sigma_{v}^{2}\sigma_{e}^{2}}{\sigma_{v}^{2} + \sigma_{e}^{2}} \Delta t.$$
(122)

The value function of the informed trader is quadratic for all t = 1, ..., T - 1:

$$\pi_t = \alpha_{t-1} (v_t - q_t)^2 + \delta_{t-1}.$$
(123)

The coefficients of the optimal trading strategy and the value function satisfy

$$\beta_t \Delta t = \frac{1 - 2\alpha_t \lambda_t}{2\lambda_t (1 - \alpha_t \lambda_t)},$$

$$\alpha_{t-1} = \beta_t \Delta t (1 - \lambda_t \beta_t \Delta t) + \alpha_t (1 - \lambda_t \beta_t \Delta t)^2,$$

$$\delta_{t-1} = \alpha_t \left(\lambda_t^2 \sigma_u^2 + \mu_t^2 (\sigma_v^2 + \sigma_e^2)\right) \Delta t + \delta_t.$$
(124)

The terminal conditions are

$$\alpha_T = \delta_T = 0. \tag{125}$$

The second order condition is

$$\lambda_t (1 - \alpha_t \lambda_t) > 0. \tag{126}$$

Given Σ_0 , conditions (122)–(126) are necessary and sufficient for the existence of a linear equilibrium.

Proof. First, we show that equations (122) are equivalent to the zero profit conditions of the market maker. Second, we show that equations (124)–(126) are equivalent to the informed trader's strategy being optimal.

Zero Profit of market maker: Let us start with with the market maker's update due to the order flow at t = 1, ..., T-1. Conditional on \mathcal{I}_t^q , v_t has a normal distribution, $v_t | \mathcal{I}_t^q \sim \mathcal{N}(q_t, \Omega_t)$. The aggregate order flow at t is of the form $\Delta y_t = \beta_t (v_t - q_t) \Delta t + \Delta u_t$. Denote by

$$\Phi_t = \mathsf{Cov}(v_t - q_t, \Delta y_t) = \beta_t \Omega_t \Delta t.$$
(127)

Then, conditional on $\mathcal{I}_t^p = \mathcal{I}_t^q \cup \{\Delta y_t\}, v_t \sim \mathcal{N}(p_t, \Sigma_t)$, with

$$p_{t} = q_{t} + \lambda_{t} \Delta y_{t},$$

$$\lambda_{t} = \Phi_{t} \operatorname{Var}(\Delta y_{t})^{-1} = \frac{\beta_{t} \Omega_{t}}{\beta_{t}^{2} \Omega_{t} \Delta t + \sigma_{u}^{2}},$$

$$\Sigma_{t} = \operatorname{Var}(v_{t} - q_{t}) - \Phi_{t} \operatorname{Var}(\Delta y_{t})^{-1} \Phi_{t}' = \Omega_{t} - \frac{\beta_{t}^{2} \Omega_{t}^{2}}{\beta_{t}^{2} \Omega_{t} \Delta t + \sigma_{u}^{2}} \Delta t$$

$$= \frac{\Omega_{t} \sigma_{u}^{2}}{\beta_{t}^{2} \Omega_{t} \Delta t + \sigma_{u}^{2}}.$$
(128)

To obtain the equation for λ_t , note that the above equations for λ_t and Σ_t imply $\frac{\lambda_t}{\Sigma_t} = \frac{\beta_t}{\sigma_u^2}$. The equation for Ω_t is obtained by solving for Σ_t in the last equation of (128).

Next, consider the market maker's update at $t = 1, \ldots, T-1$ due to the signal $\Delta z_t = \Delta v_t + \Delta e_t$. From $v_{t-1} | \mathcal{I}_{t-1}^p \sim \mathcal{N}(p_{t-1}, \Sigma_{t-1})$, we have $v_t | \mathcal{I}_{t-1}^p \sim \mathcal{N}(p_{t-1}, \Sigma_{t-1} + \sigma_v^2 \Delta t)$. Denote by

$$\Psi_t = \mathsf{Cov}(v_t - p_{t-1}, \Delta z_t) = \sigma_v^2 \Delta t.$$
(129)

Then, conditional on $\mathcal{I}_t^q = \mathcal{I}_{t-1}^p \cup \{\Delta z_t\}, v_t | \mathcal{I}_t^q \sim \mathcal{N}(q_t, \Omega_t)$, with

$$q_{t} = p_{t-1} + \mu_{t} \Delta z_{t},$$

$$\mu_{t} = \Psi_{t} \operatorname{Var}(\Delta z_{t})^{-1} = \frac{\sigma_{v}^{2}}{\sigma_{v}^{2} + \sigma_{e}^{2}},$$

$$\Omega_{t} = \operatorname{Var}(v_{t} - p_{t-1}) - \Psi_{t} \operatorname{Var}(\Delta z_{t})^{-1} \Psi_{t}' = \Sigma_{t-1} + \sigma_{v}^{2} \Delta t - \frac{\sigma_{v}^{4}}{\sigma_{v}^{2} + \sigma_{e}^{2}} \Delta t$$

$$= \Sigma_{t-1} + \frac{\sigma_{v}^{2} \sigma_{e}^{2}}{\sigma_{v}^{2} + \sigma_{e}^{2}} \Delta t.$$
(130)

Thus, we prove the equation for μ_t . Note that equation (130) gives a formula for Σ_{t-1} as a function of Ω_t , and we already proved the formula for Ω_t as a function of Σ_t in (122). We therefore get Σ_{t-1} as a function of Σ_t , which is the last equation in (122).

Optimal Strategy of Informed Trader: At each t = 1, ..., T - 1, the informed trader maximizes the expected profit: $\pi_t = \max \sum_{\tau=t}^{T-1} \mathsf{E}((v_T - p_{\tau})\Delta x_{\tau})$. We prove by backward induction that the value function is quadratic and of the form given in (123): $\pi_t = \alpha_{t-1}(v_t - q_t)^2 + \delta_{t-1}$. At the last decision point (t = T - 1) the next value function is zero, i.e., $\alpha_T = \delta_T = 0$, which are the terminal conditions (125). In the induction step, if t = 1, ..., T - 1, we assume that π_{t+1} is of the desired form. The Bellman principle of intertemporal optimization implies

$$\pi_t = \max_{\Delta x} \mathsf{E}\Big((v_t - p_t)\Delta x + \pi_{t+1} \mid \mathcal{I}_t^q, v_t, \Delta v_t\Big).$$
(131)

The last two equations in (121) show that the quote q_t evolves by $q_{t+1} = q_t + m_t \Delta y_t + \mu_t \Delta z_{t+1}$. This implies that the informed trader's choice of Δx affects the trading price

and the next quote by

$$p_t = q_t + \lambda_t (\Delta x + \Delta u_t),$$

$$q_{t+1} = q_t + \lambda_t (\Delta x + \Delta u_t) + \mu_t \Delta z_{t+1}.$$
(132)

Substituting these into the Bellman equation, we get

$$\pi_{t} = \max_{\Delta x} \mathsf{E} \Big(\Delta x (v_{t} - q_{t} - \lambda_{t} \Delta x - \lambda_{t} \Delta u_{t}) \\ + \alpha_{t} (v_{t} + \Delta v_{t+1} - q_{t} - \lambda_{t} \Delta x - \lambda_{t} \Delta u_{t} - \mu_{t} \Delta z_{t+1})^{2} + \delta_{t} \Big)$$

$$= \max_{\Delta x} \Delta x (v_{t} - q_{t} - \lambda_{t} \Delta x) \\ + \alpha_{t} \Big((v_{t} - q_{t} - \lambda_{t} \Delta x)^{2} + (\lambda_{t}^{2} \sigma_{u}^{2} + \mu_{t}^{2} (\sigma_{v}^{2} + \sigma_{e}^{2})) \Delta t \Big) + \delta_{t}.$$
(133)

The first order condition with respect to Δx is

$$\Delta x = \frac{1 - 2\alpha_t \lambda_t}{2\lambda_t (1 - \alpha_t \lambda_t)} (v_t - q_t), \qquad (134)$$

and the second order condition for a maximum is $\lambda_t(1 - \alpha_t \lambda_t) > 0$, which is (126). Thus, the optimal Δx is indeed of the form $\Delta x_t = \beta_t (v_t - q_t) \Delta t$, where $\beta_t \Delta t$ satisfies equation (124). We substitute Δx_t in the formula for π_t to obtain

$$\pi_t = \left(\beta_t \Delta t (1 - \lambda_t \beta_t \Delta t) + \alpha_t (1 - \lambda_t \beta_t \Delta t)^2\right) (v_t - q_t)^2 + \alpha_t \left(\lambda_t^2 \sigma_u^2 + \mu_t^2 (\sigma_v^2 + \sigma_e^2)\right) \Delta t + \delta_t.$$
(135)

This proves that indeed π_t is of the form $\pi_t = \alpha_{t-1}(v_t - q_t)^2 + \delta_{t-1}$, with α_{t-1} and δ_{t-1} as in (124).

Equations (122) and (124) form a system of equations. As before, it is solved backwards, starting from the boundary conditions (125), and so that $\Sigma_t = \Sigma_0$ at t = 0.

C Sampling at Lower Frequency than Trading Frequency

Suppose trades are aggregated over short time intervals of length $\Delta \tau$. Then, data are indexed by $\tau = 0, 1, 2, \dots, \frac{1}{\Delta \tau} - 1$ and the informed order flow at τ is $\Delta x_{\tau} = x_{\tau \Delta \tau + \Delta \tau} - x_{\tau \Delta \tau}$. The empirical counterpart of the Informed Participation Rate and the autocorrelation of the informed investor's order flow when data are sampled every $\Delta \tau$ periods of time are, respectively,

$$IPR(\Delta\tau) = \frac{\operatorname{Var}(\Delta x_{\tau}(\Delta\tau))}{\operatorname{Var}(\Delta x_{\tau}(\Delta\tau)) + \operatorname{Var}(\Delta u_{\tau}(\Delta\tau))},$$

$$\operatorname{Corr}(\Delta x_{\tau}(\Delta\tau), \Delta x_{\tau+k}(\Delta\tau)).$$
(136)

Proposition 13. When $\Delta \tau$ is small, the empirical informed participation rate in the benchmark increases with the sampling interval $\Delta \tau$ and is always below its level in the fast model:

$$IPR^{B}(\Delta\tau) = \frac{(\beta_{t}^{B})^{2}\Sigma_{t}}{\sigma_{u}^{2}}\Delta\tau + o(\Delta\tau),$$

$$IPR^{F}(\Delta\tau) = \frac{(\gamma^{F})^{2}\sigma_{v}^{2}}{(\gamma^{F})^{2}\sigma_{v}^{2} + \sigma_{u}^{2}} + o(1).$$
(137)

The informed order flow autocorrelation in the fast model increases with the sampling interval $\Delta \tau$ and is always below its level in the benchmark:

$$\operatorname{Corr}(\Delta x_{\tau}^{B}(\Delta \tau), \Delta x_{\tau+k}^{B}(\Delta \tau)) = \left(\frac{1-t-k\Delta\tau}{1-t}\right)^{\lambda^{B}\beta_{0}^{B}-\frac{1}{2}} + o(1),$$

$$\operatorname{Corr}(\Delta x_{\tau}^{F}(\Delta \tau), \Delta x_{\tau+k}^{F}(\Delta \tau)) = \frac{\beta_{t+k\Delta\tau}^{F}(\beta_{t}^{F}\Sigma_{t}+\gamma^{F}(1-m^{F}\gamma^{F}-\mu^{F})\sigma_{v}^{2})}{(\gamma^{F})^{2}\sigma_{v}^{2}} \qquad (138)$$

$$\times \left(\frac{1-t-k\Delta\tau}{1-t}\right)^{m^{F}\beta_{0}^{F}}\Delta\tau + o(\Delta\tau).$$

To define the empirical counterpart of our measure of anticipatory trading, we now consider that trading takes place in discrete time rather than in continuous time. The infinitesimal time interval dt is replaced by a real number $\Delta t > 0$. Time is thus indexed by $t = 0, 1, 2, \ldots, \frac{1}{\Delta t} - 1$. We assume that Δt is small and we approximate the equilibrium variables $(\beta_t, \gamma_t, \lambda_t, \mu_t, \rho_t)$ in this discrete time model by their continuous time counterpart. The informed trade at time t is equal to $\Delta x_t = \beta_t (v_t - q_t) \Delta t + \gamma \Delta v_t$, where q_t is the quote just before the order flow arrives, and p_{t+1} is the execution price.

We consider that the econometrician has data sampled every n trading rounds, i.e., the sampling interval is $\Delta \tau = n \Delta t$. Therefore the data are a time-series indexed by $\tau = 0, 1, 2, \dots, \frac{1}{\Delta \tau} - 1$ and the τ -th observation corresponds to trading during the ntrading rounds from $t = \tau n$ to $t = \tau n + n - 1$:

$$\Delta x_{\tau}(\Delta \tau) = \Delta x_{\tau n} + \ldots + \Delta x_{\tau n+n-1}, \qquad (139)$$

and the return over this period is:

$$r_{\tau}(\Delta \tau) = p_{t+n} - p_t. \tag{140}$$

Finally, consider the empirical counterpart of our measure of anticipatory trading:

$$AT_{\tau}(\Delta \tau) = \operatorname{Corr}(\Delta x_{\tau}(\Delta \tau), r_{\tau+1}(\Delta \tau)).$$
(141)

The next result shows that that sampling data at a sufficiently high frequency (i.e., low $\Delta \tau$) is important for detecting anticipatory trading.

Proposition 14. In the fast model, the empirical measure of anticipatory trading decreases with $\Delta \tau$ and converges to zero when $\Delta \tau \to +\infty$:

$$AT_{\tau}^{F}(\Delta\tau) = \frac{\mu^{F}(1-\rho^{F}\gamma^{F})\sigma_{v}}{\sqrt{\sigma_{v}^{2}+\Sigma_{0}}}\frac{\Delta t}{\Delta\tau}.$$
(142)

The aggregated order flow spans $n = \frac{\Delta \tau}{\Delta t}$ trading periods. Moreover, each trade anticipates news that is incorporated in the quotes in the next trading round. Therefore, only the last trade of the aggregated order flow $\Delta x_{\tau}(\Delta \tau)$ is correlated with the next aggregated return $r_{\tau+1}(\Delta \tau)$. As a result, when *n* increases, the correlation between $\Delta x_{\tau}(\Delta \tau)$ and $r_{\tau+1}(\Delta \tau)$ decreases. When *n* becomes too large, the correlation becomes almost zero.

C.1 Proof of Proposition 13

Denoting $t = \tau \Delta \tau$, the τ th trade in the data is $\Delta x_{\tau} = x_{t+\Delta\tau} - x_t = \int_{u=t}^{t+\Delta\tau} \beta_u (v_u - p_u) du + \gamma dv_u$. When $\Delta \tau$ is small, in the benchmark model we have $\operatorname{Var}(\Delta x_{\tau}^B) = (\beta_t^B)^2 \Sigma_t (\Delta \tau)^2 + o((\Delta \tau)^2)$, which yields the informed participation rate in (137). Using Lemma 2, we obtain: $\operatorname{Cov}(\Delta x_{\tau}^B, \Delta x_{\tau+k}^B) = \beta_{t+k\Delta\tau}^B \beta_t^B \Sigma_t \left(\frac{1-t-k\Delta\tau}{1-t}\right)^{\lambda^B \beta_0^B} (\Delta \tau)^2 + o((\Delta \tau)^2)$, which proves the first equation in (138).

In the fast model: $\operatorname{Var}(\Delta x_{\tau}^{F}) = (\gamma^{F})^{2} \sigma_{v}^{2} \Delta \tau + o(\Delta \tau)$, from which follows the informed participation rate in (137). Using Lemma 2, $\operatorname{Cov}(\Delta x_{\tau}^{F}, \Delta x_{\tau+k}^{F}) = \beta_{t+k\Delta\tau}^{F}(\beta_{t}^{F}\Sigma_{t} + \gamma^{F}(1-m^{F}\gamma^{F}-\mu^{F})\sigma_{v}^{2}) \left(\frac{1-t-k\Delta\tau}{1-t}\right)^{m^{F}\beta_{0}^{F}} (\Delta \tau)^{2} + o((\Delta \tau)^{2})$, where $m^{F} = \lambda^{F} - \mu^{F}\rho^{F}$, which yields the second equation in (138).

C.2 Proof of Proposition 14

When $\Delta \tau$ is small, we have $\operatorname{Cov}(\Delta x_{\tau}(\Delta \tau), r_{\tau+1}(\Delta \tau)) = \gamma^F \mu^F (1 - \rho^F \gamma^F) \sigma_v^2 \Delta t$. Also, $\operatorname{Var}(\Delta x_{\tau}) = (\gamma^F)^2 \sigma_v^2 \Delta \tau + o(\Delta \tau)$, and $\operatorname{Var}(r_{\tau+1}) = (\sigma_v^2 + \Sigma_0) \Delta \tau + o(\Delta \tau)$. These prove (142).

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