Portfolio Theory with a Drift

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1 Introduction

The validity of the Markowitz approach to portfolio management, i.e., the mean/variance view on risk and return and, as a consequence, the validity of the CAPM have been questioned time and again in the literature. Most of these investigations focus on the underlying assumptions being not true in the real world. Specifically, asset returns are not normally distributed and neither volatilities nor correlations are constant over any reasonable holding period δt and *therefore* the volatility (which is the heart of the Markowitz theory) is not a suitable risk measure.

Rather then re-stating all these investigations of empirical evidence for or against Markowitz and the CAPM, we take a different approach here. We stay *within* the Markowitz framework (which is basically the same as the Black-Scholes framework), i.e., we hold on to the assumptions that asset returns are normally distributed with constant volatilities and correlations and show, that *even within this framework* the volatility is not a suitable risk measure. We thus beat Markowitz theory with its own weapons, so to say.

2 Markowitz in a Nut Shell

The goal of modern portfolio management is to optimize *risk adjusted performance measures* (abbreviated "*RAPM*"), i.e., ratios of the kind

$$\frac{\text{(expected) portfolio return}}{\text{portfolio risk}} \tag{2.1}$$

In Markowitz theory the risk of a portfolio V consisting of holdings N_i in M risky assets (the *risk factors*) with values V_i, volatilities σ_i, expected returns R_i and mutual correlations ρ_{ij} is simply the portfolio *volatility*

$$\sigma_V = \sqrt{\mathbf{w}^T \mathbf{C} \mathbf{w}} \tag{2.2}$$

Here $\mathbf{C}$ denotes the covariance matrix with the elements $C_{ij} = \sigma_i \rho_{ij} \sigma_j$ for $i, j = 1, \ldots M$ and $\mathbf{w}$ is the vector of asset weights, i.e. $w_i = N_i V_i / V$ for $i = 1, \ldots M$.

As explained in most texts on portfolio theory, the optimal investment strategy (yielding the highest expected return for the risk incurred) within Markowitz theory is to invest in the so called *Market Portfolio*. This portfolio is the fully invested portfolio having maximum *Sharpe Ratio*

$$\gamma_m = \frac{\widehat{R}_m}{\sigma_m} \tag{2.3}$$

where σ_m denotes the volatility of the market portfolio and $\widehat{R}_m := R_m - r_f$ denotes its expected *excess* return[1] (above the risk free rate r_f).

Even if the risk σ_m of the market portfolio doesn't coincide with the risk preference σ_{required} of an investor, the investor should still invest in this portfolio, although not all of his money. If $\sigma_{\text{required}} < \sigma_m$ the investor should only invest a percentage w of the total capital in the market portfolio and the rest of the capital should be invested risk free (in a money market account). On the other hand, if $\sigma_{\text{required}} > \sigma_m$ the investor should borrow money (from the money market) and invest the total sum of his own capital and the loan in the market portfolio (*leveraged investment*).

Since the optimal portfolio has the maximum excess return the investor can expect for the risk taken and since investing or borrowing in the money market produces neither any additional excess return nor any additional volatility, this strategy gives the best possible Sharpe Ratio for any required risk level. The expected return of this strategy as a function of the risk (the volatility) incurred is a straight line with slope γ_m, the *Capital Market Line*.

Within this framework all sorts of portfolio optimizations can be done, even analytically as long as no constraints (like for instance "no short selling" or the like) have to be obeyed. For example, asset weights, excess return and volatility of the market portfolio can be calculated (Deutsch (2004))

$$\mathbf{w}_m = \frac{\mathbf{C}^{-1}\widehat{\mathbf{R}}}{\mathbf{1}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}, \quad \widehat{R}_m = \frac{\widehat{\mathbf{R}}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}{\mathbf{1}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}, \quad \sigma_m^2 = \frac{\widehat{R}_m}{\mathbf{1}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}} \tag{2.4}$$

where $\widehat{\mathbf{R}}$ is the vector of the asset's expected excess returns R_i with components $\widehat{R}_i := R_i - r_f$ for $i = 1, \dots M$.

3 A Better Risk Measure

To compare classical Markowitz theory with modern risk concepts, we have to start all over again, and carefully ask: What is risk? Very generally, *risk* is defined as a *potential loss*, which will not be exceeded over a certain time horizon δt (the *holding period*) with a specified probability c (the *confidence*), see for instance (Deutsch (2004)), (Jorion (1997)) or (RiskMetrics (1996)). In the world of normally distributed asset returns this leads (Deutsch (2004)) to the *Value at Risk* of the above mentioned portfolio V consisting of the M risky assets (the risk factors)

$$\mathrm{VaR}_{c,\delta t}(R_V) \approx |Q_{1-c}|\, V\sqrt{\delta t}\sqrt{\sum_{k,l=1}^{M} w_k\sigma_i\rho_{ij}\sigma_j w_l} - V\delta t \sum_{k=1}^{M} w_k R_k$$

where Q_{1-c} denotes the $(1-c)$ percentile of the standard normal distribution. This can be written more compactly using obvious vector notation:

$$\begin{aligned}\mathrm{VaR}_V(c) &\approx |Q_{1-c}|\, V\sqrt{\delta t}\sqrt{\mathbf{w}^T\mathbf{C}\mathbf{w}} - V\delta t\, \mathbf{w}^T\mathbf{R} \\ &= |Q_{1-c}|\, V\sqrt{\delta t}\sigma_V - V\delta t\, R_V\end{aligned} \tag{3.1}$$

with the covariance matrix C having the matrix elements $C_{ij} = \sigma_i\rho_{ij}\sigma_j$ for $i, j = 1, \dots M$ as in Eq. 2.2.

The generalization to the case where the assets in the portfolio are not the risk factors themselves but M financial instruments with values V_k depending on n risk factors S_i is straight forward. As shown in (Deutsch (2004)), the Delta-Normal approximation of the VaR for this situation looks exactly the same as Eq. 3.1, but $\mathbf{C}$ now is the covariance matrix of the *financial instruments* which (in first order approximation) is related to the covariances of the *risk factors* via

$$C_{kl} = \sum_{i,j=1}^{n} \Omega_i^k \sigma_i\rho_{ij}\sigma_j\Omega_j^l \quad \text{with} \quad \Omega_i^k = \frac{S_i}{V_k}\Delta_i^k$$

where $\Delta_i^k = \partial V_k/\partial S_i$ denotes the linear price sensitivity (the *Delta*) of the k^{th} financial instrument with respect to the i^{th} risk factor. To arrive at this Delta-Normal Value at Risk, several approximations have to be made (in addition to all assumptions of Markowitz theory), namely

1. The functional dependencies of the instrument prices on the risk factors are approximated linearly
2. The exponential risk factor evolution (i.e., the solutions to the stochastic differential equations describing the risk factor processes as correlated Brownian motions, see (Deutsch (2004))) is approximated linearly.

However, besides these approximations and besides all the assumption of normally distributed returns and constant volatilities and correlations, it is still not possibly to bring the VaR in Eq. 3.1 in line with the even simpler risk definition of Markowitz theory, where the portfolio *risk* is simply the portfolio *volatility* as in Eq.2.2. For this, we need one more crucial approximation, namely

> The risk factor drifts are neglected since (for reasonably short holding periods) the risk resulting from fluctuations is much larger than the effect of the drift (Riskmetrics (1996)).

This would neglect the second term in Eq. 3.1 and the VaR would then indeed be proportional to the volatility. However, while the first two approximations are usually acceptable for portfolios with rather linear assets (not for options portfolios, however) and for not too long holding periods, the neglect of the drift is in direct contradiction to the very concept of the Capital Market Line describing the optimal investment strategy! As explained above, an investor can achieve *any* desired volatility with this strategy by distributing money between the market portfolio and the (risk free) money market account. In particular, he can attain *arbitrarily small* volatilities. But an arbitrarily small volatility is not large compared to the drift! Therefore, **the drift must not be neglected (not even for short holding periods) as soon as the money market comes into play!** We have to stick with Eq. 3.1 as our risk measure and cannot reduce it to Eq. 2.2. The consequences of this circumstance are far reaching and are the topic of this paper.

3.1 Being not fully invested

Let's now look at the risk of a typical investment on the Capital Market Line, where part of the money is invested in (or borrowed from) the money market. In this case we have a percentage

$$w = \sum_{i=1}^{M} w_i = \mathbf{w}^T\mathbf{1}$$

invested in the risky assets V_i and a part $w_f = 1 - w$ invested in the risk free account (for a leveraged investment we have $w_f < 0$ and $w > 1$). Of course, the risk free part does not contribute to the fluctuations of the total investment. It does, however, contribute to the expected return, i.e. to the drift of the total investment. Adding the risk free contribution $w_f r_f$

to the drift yields

$$\begin{aligned}\mathrm{VaR}_V(c) &\approx |Q_{1-c}|\, V\sqrt{\delta t}\sqrt{\mathbf{w}^T\mathbf{C}\mathbf{w}} - V\delta t\left[\mathbf{w}^T\mathbf{R} + w_f r_f\right]\\ &= |Q_{1-c}|\, V\sqrt{\delta t}\sqrt{\mathbf{w}^T\mathbf{C}\mathbf{w}} - V\delta t\left[\mathbf{w}^T\mathbf{R} + (1-\mathbf{w}^T\mathbf{1})r_f\right]\\ &= |Q_{1-c}|\, V\sqrt{\delta t}\sqrt{\mathbf{w}^T\mathbf{C}\mathbf{w}} - V\delta t\left[\mathbf{w}^T(\mathbf{R}-\mathbf{1}r_f) + r_f\right]\end{aligned}$$

Thus, the Value at Risk for the total investment is

$$\begin{aligned}\mathrm{VaR}_V(c) &\approx |Q_{1-c}|\, V\sqrt{\delta t}\sqrt{\mathbf{w}^T\mathbf{C}\mathbf{w}} - V\delta t[\mathbf{w}^T\widehat{\mathbf{R}} + r_f]\\ &= |Q_{1-c}|\, V\sqrt{\delta t}\,\sigma_V - V\delta t\left[\widehat{R}_V + r_f\right]\\ &= |Q_{1-c}|\, V\sqrt{\delta t}\,\sigma_V - V\delta t R_V\end{aligned} \tag{3.2}$$

Here $R_V \equiv \mathrm{E}[r_V]$ is the expected return of the total investment and $\widehat{R}_V$ is the expected *excess* return of the total investment. Thus, we get back Equation 3.1 also for not fully invested portfolios.

Observe that the percentile Q_{1-c} of the normal distribution (depending on the required confidence level c) and the holding period δt do not enter as common factors. This should also intuitively be clear: the smaller the quantile (the larger the confidence) the larger is the fluctuations' influence on the VaR compared to the drift. Similarly: the larger the holding period, the larger is the drift effect compared to the fluctuations. The relative influences of drift and fluctuations on the risk depend strongly on the chosen confidence level c and holding period δt. Therefore, c and δt can *not* be eliminated from the risk definition.

We now divide the VaR by V to get a risk measure in percentage terms, i.e., independent of the total amount invested.We also divide by δt to generate dimensionless RAPMs of the form like Equation 2.1. Another motivation for dividing by $V\delta t$ is to write the VaR directly in terms of the annualized portfolio returns r_V:

$$\mathrm{VaR}_V(c) \approx |Q_{1-c}|\, V\delta t\sqrt{\mathrm{var}[r_V]} - V\delta t\,\mathrm{E}[r_V] \tag{3.3}$$

This form can be derived by observing that[2]

$$\sigma_V^2 \equiv \frac{1}{\delta t}\mathrm{var}[\delta\ln(V)] = \delta t\,\mathrm{var}[r_V]$$

In this form, $V\delta t$ even looks like a common factor[3]. Thus, our risk measure replacing the volatility is the risk *per unit of time and per monetary unit invested*

$$\begin{aligned}\eta_V &\equiv \frac{\mathrm{VaR}_V(c)}{V\,\delta t}\\ &= |Q_{1-c}|\sqrt{\mathrm{var}[r_V]} - \mathrm{E}[r_V]\\ &= q\sigma_V - R_V\\ &= q\sqrt{\mathbf{w}^T\mathbf{C}\mathbf{w}} - \mathbf{w}^T\widehat{\mathbf{R}} - r_f\end{aligned} \tag{3.4}$$

where we have introduced the abbreviation

$$q \equiv |Q_{1-c}|/\sqrt{\delta t} \tag{3.5}$$

to streamline the notation. Keep in mind that q is not an *overall* constant and therefore can *not* be ignored as in classical Markowitz theory.

In addition, compared to Equation 2.2, we now have the term $-R_V$ taking the drift's influence on the risk (defined as the potential loss occurring with probability $1-c$) into account. This produces an important effect: The expected investment return (which includes the risk free earnings) influences the risk. Therefore, whenever one is not fully invested, **the risk free money market influences the investment risk,** since it contributes to the investment's expected return. This has severe consequences. The most important consequence is the fact, that Markowitz theory can *not* be "saved" by simply replacing σ with η everywhere. As we will now see, things are more subtle.

4 The Capital Market Line with Drift

Although we haven't constructed it yet, let's assume that for any given *investment universe* (i.e., for any given set of M risky assets) there is a fully invested *optimal portfolio* with return R_m and risk η_m. As in Markowitz theory, if the risk η_m of the optimal portfolio doesn't coincide with the risk preference η_{required} of the investor, the investor still invests in the optimal portfolio, although not all of his money. If $\eta_{\text{required}} < \eta_m$ the investor only invests a percentage w of the total capital in the optimal portfolio and the rest of the capital in a risk free money market account. On the other hand, if $\eta_{\text{required}} > \eta_m$ the investor borrows money from the money market and invests the total sum of his own capital and the loan in the optimal portfolio.

The expected return of such an investment in the money market account and the optimal portfolio is

$$R_V = wR_m + (1-w)r_f \tag{4.1}$$

where $w := \mathbf{w}^T\mathbf{1}$ denotes the part invested in the risky assets. With this R_V, the risk, as defined in Equation 3.4, of such an investment is

$$\begin{aligned}\eta_V &= q\sigma_V - R_V\\ &= q\sigma_V - wR_m - (1-w)r_f\end{aligned}$$

which again explicitly shows how the risk free rate influences the risk of the total investment. We can write $\sigma_V = w\sigma_m$ since the risk free return has no volatility[4] to arrive at

$$\begin{aligned}\eta_V &= w(q\sigma_m - R_m) - (1-w)r_f\\ &= w\eta_m - (1-w)r_f\end{aligned} \tag{4.2}$$

This should also intuitively be clear: The part w of the investment in the optimal portfolio contributes the risk of the optimal portfolio while the part $(1-w)$ invested in the money market reduces the risk by its expected (risk free) return.

Solving Equation 4.2 for the leverage w, we find that (in contrast to Markowitz theory) the extent of investment in risky asset is not simply given by the ratio of the investment risk to the risk of the optimal portfolio, but rather by

$$w = \frac{\eta_V + r_f}{\eta_m + r_f} \tag{4.3}$$

Inserting this w into Equation 4.1 allows us to write the investment return as a function of the investment risk

$$\begin{aligned} R_V &= \frac{\eta_V + r_f}{\eta_m + r_f} R_m + (1 - \frac{\eta_V + r_f}{\eta_m + r_f})\, r_f \\ &= \frac{\eta_V + r_f}{\eta_m + r_f} R_m + \frac{\eta_m - \eta_V}{\eta_m + r_f}\, r_f \\ &= \frac{\left(R_m - r_f\right) \eta_V + \left(R_m + \eta_m\right) r_f}{\eta_m + r_f} \end{aligned}$$

or

$$R_V = \frac{R_m - r_f}{\eta_m + r_f} \cdot \eta_V + \frac{\eta_m + R_m}{\eta_m + r_f} \cdot r_f \qquad (4.4)$$

Thus, as in Markowitz theory, the expected investment return R_V as a function of the investment risk η_V is a straight line. This straight line is the *capital market line* when drift effects are considered. However, the slope of this line is *not* given by the Sharpe Ratio of Eq. 2.3 but rather by

$$\Gamma_m \equiv \frac{R_m - r_f}{\eta_m + r_f} = \frac{\widehat{R}_m}{\eta_m + r_f} = \frac{\widehat{R}_m}{q\sigma_m - \widehat{R}_m} \qquad (4.5)$$

We call this ratio[5] the *Deutsch Ratio*™. Just like the Sharpe Ratio, the Deutsch Ratio™ is not only defined for the optimal portfolio V_m but for any portfolio[6]:

$$\Gamma_V \equiv \frac{R_V - r_f}{\eta_V + r_f} = \frac{\widehat{R}_V}{q\sigma_V - \widehat{R}_V} \qquad (4.6)$$

5 The Risk of the *Excess* Returns

The numerator in the Deutsch Ratio™ (and in the Sharpe Ratio for that matter) has a very natural interpretation: it is the expected value of the investment's *excess* returns. To achieve a just as natural interpretation for the denominator of the Deutsch Ratio™ observe the following: The Value at Risk in Equation 3.2 or 3.3 is the quantile of the distribution of the investment returns r_V. Since the investment's *excess* returns $\widehat{r}_V$ are obtained by simply subtracting the *constant* risk free rate r_f from *each* r_V, the distribution of the *excess* returns $\widehat{r}_V$ has exactly the same shape as the distribution of the returns r_V, simply shifted by $-r_f$. Thus, the Value at Risk of the *excess* returns, defined in exactly the same way as the "usual" VaR, i.e. as the quantile of their distribution, is

$$\mathrm{VaR}_c(\widehat{r}_V) = \mathrm{VaR}_c(r_V) + V\,\delta t\, r_f \qquad (5.1)$$

This can of course also be seen directly from Equation 3.3 by observing that r_f only adds to the expected return but leaves the variance unchanged:

$$\begin{aligned} \mathrm{VaR}_c(\widehat{r}_V) &= |Q_{1-c}|\, V\delta t \sqrt{\mathrm{var}\left[\widehat{r}_V\right]} - V\delta t\, \mathrm{E}\left[\widehat{r}_V\right] \\ &= |Q_{1-c}|\, V\delta t \sqrt{\mathrm{var}\left[r_V\right]} - V\delta t\, \mathrm{E}\left[r_V - r_f\right] \\ &= \underbrace{|Q_{1-c}|\, V\delta t \sqrt{\mathrm{var}\left[r_V\right]} - V\delta t\, \mathrm{E}\left[r_V\right]}_{\mathrm{VaR}_c(r_V)} + V\delta t\, r_f \end{aligned}$$

Dividing by $V\,\delta t$ as in Equation 3.4 we get the risk measure $\widehat{\eta}$ of the *excess* returns in the same units as our risk measure η

$$\begin{aligned} \widehat{\eta}_V &\equiv \frac{\mathrm{VaR}_c(\widehat{r}_V)}{V\delta t} \\ &= |Q_{1-c}| \sqrt{\mathrm{var}\left[\widehat{r}_V\right]} - \mathrm{E}\left[\widehat{r}_V\right] \\ &= q\sigma_V - \widehat{R}_V \\ &= q\sqrt{\mathbf{w}^T\mathbf{C}\mathbf{w}} - \mathbf{w}^T\widehat{\mathbf{R}} \\ &= \eta_V + r_f \end{aligned} \qquad (5.2)$$

Thus, the denominator of the Deutsch Ratio™ is the *risk of the excess returns*. And the Deutsch Ratio™ itself is simply the expected excess return divided by the risk of the excess returns:

$$\Gamma_V \equiv \frac{\widehat{R}_V}{\widehat{\eta}_V} = \frac{\text{expected excess return}}{\text{risk of excess returns}} \qquad (5.3)$$

As already known for the expected returns, we now also see for the *risk* that *excess* returns (instead of returns) are the most natural quantities to consider in asset management.

In asset management it is much more natural to *always* work with excess returns, not only regarding expectations, but also regarding *risk*.

As an example of how much more natural excess returns are, observe the following: The investment risk η_V in Eq. 4.2 is not zero for $w = 0$ but rather for

$$w_0 := \frac{r_f}{\eta_m + r_f} \qquad (5.4)$$

as can easily be verified by inserting w_0 into Eq. 4.2. A leverage $w < w_0$ has "negative risk". This means that the fluctuations of the investment value are so small compared to the (positive) drift, that the investment return at the border of the confidence interval (i.e. the quantile of the return distribution belonging to confidence c) is still positive. Or in other words: the potential loss incurred by the fluctuations is still less than the expected return.

For instance, we have for $w = 0 < w_0$ from Equations 4.1 and 4.2 $R_V = r_f$ and $\eta_V = -r_f$. This is fully consistent: For $w = 0$ the whole investment is placed in the money market. Thus, the expected return is the money market return without any fluctuations and the risk for any confidence level, i.e. the potential loss, is the negative of the one and only P&L value r_f. However, it might seem awkward that the so called risk *free* investment should have a non-zero risk (and a negative one at that!). This does *not* happen, if one uses the risk of the *excess* returns, Equation 5.2, as the risk measure: Since $\widehat{\eta}_V = \eta_V + r_f$, this risk measure is exactly zero when everything is invested risk free.

As another example look at the second term in Eq. 4.4 . The expected return as a function of investment risk in Eq. 4.4 has on *offset*. For risk $\eta_V = 0$ (attainable through the leverage $w = w_0$ according to Eq. 5.4) the expected return is

$$R_V = \frac{\eta_m + R_m}{\eta_m + r_f} r_f$$

>

which is larger than the risk free rate r_f as long as the return R_m of the optimal portfolio is larger than r_f (which usually is the case as a compensation for the risk of the optimal portfolio). Thus, even for $\eta_V = 0$ the expected return is larger than the risk free return. This is the compensation for the risk of incurring a loss greater than the border of the confidence interval belonging to the chosen confidence c. As already stressed, $\eta_V = 0$ only means that the negative influence of the fluctuations just compensates the positive influence of the drift. But it does not mean that there are no fluctuations[7]!

All these "hard to digest" facts disappear, as soon as one uses excess returns throughout. The capital market line in terms of *excess* returns can be derived by simply subtracting r_f from Equation 4.4:

$$\begin{aligned}\widehat{R}_V &= R_V - r_f \\ &= \frac{R_m - r_f}{\eta_m + r_f}\cdot\eta_V + \frac{\eta_m + R_m - (\eta_m + r_f)}{\eta_m + r_f}\cdot r_f \\ &= \frac{R_m - r_f}{\eta_m + r_f}\cdot(\eta_V + r_f)\end{aligned}$$

Thus, the Capital Market line can be written very elegantly as

$$\widehat{R}_V = \Gamma_m\widehat{\eta}_V \quad \text{with} \quad \Gamma_m = \frac{\widehat{R}_m}{\widehat{\eta}_m} \tag{5.5}$$

No more offset! The expected excess return is zero when the excess return risk is zero. And - as discussed above - the excess return risk is zero when everything is invested risk free.

5.1 The Condition for Positive Risk of Excess Returns

But even the *excess* return risk $\widehat{\eta}_V$ in Eq. 5.2 could become negative, namely when the confidence c is chosen so low[8] and/or the holding period δt so large that $q = |Q_{1-c}|/\sqrt{\delta t}$ is smaller than $\widehat{R}_V/\sigma_V$. The latter happens to be the traditional Sharpe Ratio γ. Thus, we have the following requirement for a sensible choice of parameters, i.e., parameters which guarantee positive risk of excess returns:

$$\frac{|Q_{1-c}|}{\sqrt{\delta t}} \equiv q \overset{!}{>} \gamma_V \equiv \frac{\widehat{R}_V}{\sigma_V} \tag{5.6}$$

To see what this means for the available choices of confidence c, given a holding period δt, observe that the $1-c$ quantile of the standard normal distribution is negative for any confidence $c > 50\%$, i.e., for any reasonable confidence level we have

$$|Q_{1-c}| = -Q_{1-c} \quad \forall c > 50\%$$

For such confidence levels we therefore obtain the requirement

$$\begin{aligned} -Q_{1-c} &\overset{!}{>} \frac{\widehat{R}_V}{\sigma_V}\sqrt{\delta t} \\ 1 - c &< \mathrm{N}\left(-\frac{\widehat{R}_V}{\sigma_V}\sqrt{\delta t}\right) \\ c &> 1 - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\sqrt{\delta t}\,\widehat{R}_V/\sigma_V} e^{-x^2/2}dx \end{aligned} \tag{5.7}$$

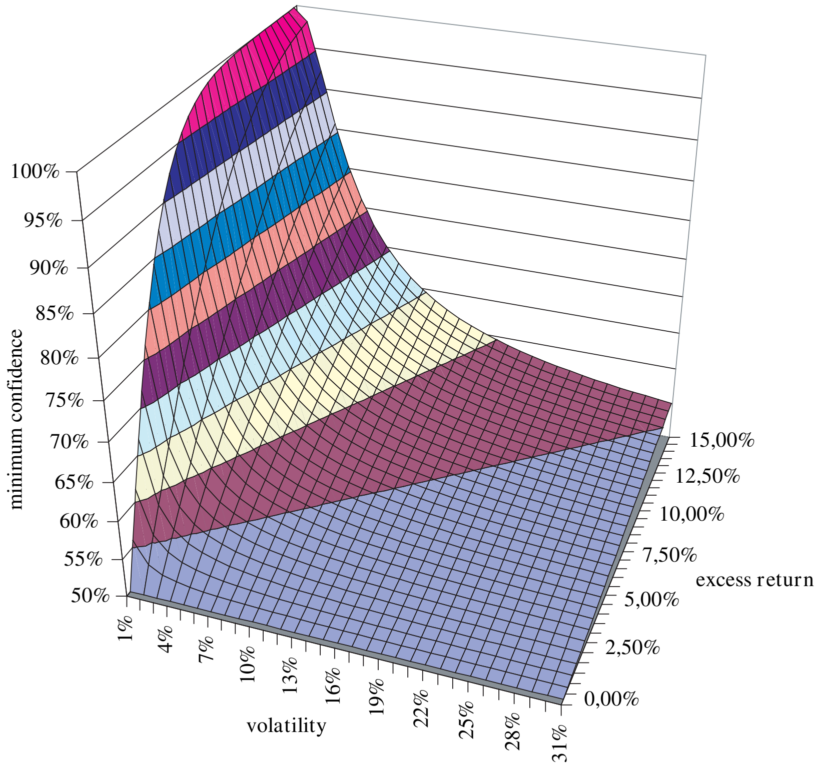

Figure 5.1: Minimum required confidence level to guarantee positive excess return risk for a portfolio with a holding period of 30 days as a function of the portfolio's volatility and expected excess return.

Numerical examples show, that this is not much of a restriction in practice, see Figure 5.1. Only portfolios with very high expected excess return and *simultaneously* very low volatility would require a confidence c significantly larger than ca. 70%. Such portfolios are unrealistic, however, since for such low volatilities the expected portfolio return should approach the risk free return, i.e. the expected *excess* return should approach zero.

For confidence levels above the lower bound required by Eq. 5.7, portfolio optimization based on maximizing the Deutsch Ratio™ works very well, since the Deutsch Ratio™ behaves nicely, see Figure 5.2.

If the confidence is chosen to low, however, the risk of excess returns (i.e. the denominator of the Deutsch Ratio™) goes from positive values through zero to negative values and generates artificial poles of the Deutsch Ratio™ leading to spurious optimization results strongly dependent on the parameter choice, see Figure 5.3. One should therefore always make sure that the confidence is above the required minimum level when performing portfolio optimization based on the Deutsch Ratio™.

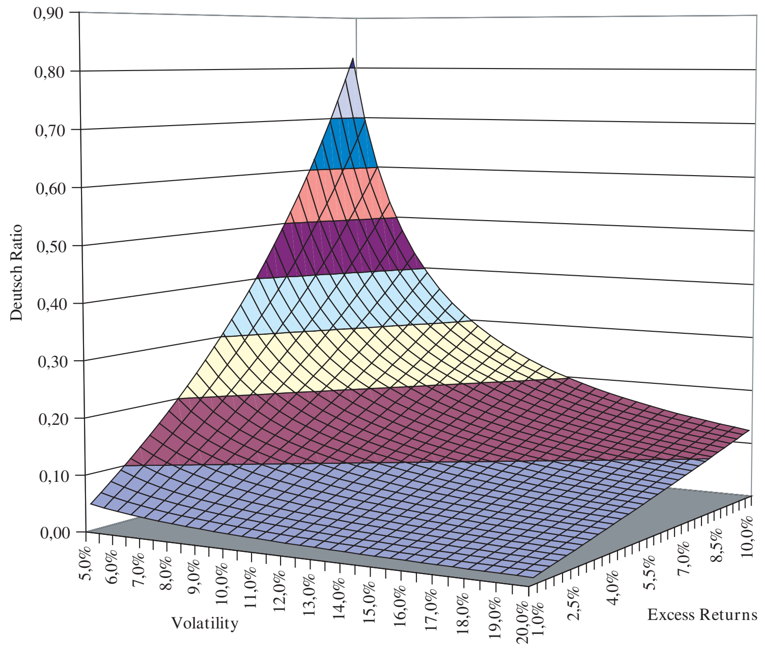

Figure 5.2: Deutsch Ratio for 90% confidence of a portfolio with a holding period of 30 days as a function of the portfolio's volatility and expected excess return. For the most extreme combination of volatility and excess return in the picture ($\widehat{R}_V = 10\%,\ \sigma_V = 5\%$), the minimum required confidence level according to Eq. 5.7 is 71,8%.

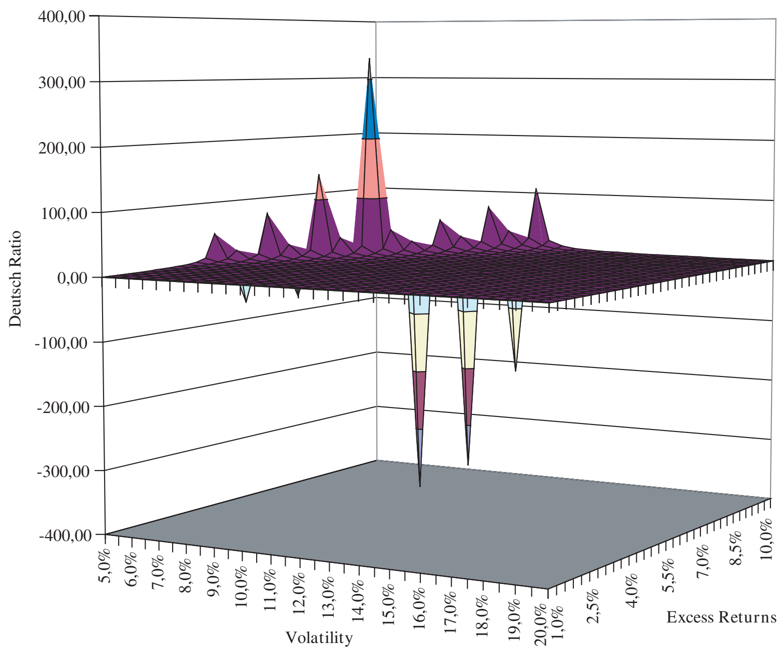

Figure 5.3: Deutsch Ratio for the same situation as in figure 5.2. The only difference is that instead of 90% the confidence was now chosen to be $c = 60\%$ which is well below the 71,8% required by Eq. 5.7.

6 Interpretation of the Deutsch Ratio™

The most intuitive interpretation of the Deutsch Ratio™ is already given by Equation 5.3 above, i.e., the Deutsch Ratio™ is the ratio of the expectation to the risk of the *excess* returns. In addition, there are some more insights worth mentioning.

6.1 The Deutsch Ratio™ and the Market Price of Risk

Equation 5.5 can be read in the following way: For each unit of additional risk $\widehat{\eta}_V$ an investor is willing to take, the expected return of the total investment increases by an amount Γ_m, i.e., by the Deutsch Ratio™ of the optimal portfolio[9]. Γ_m is therefore the *market price of risk* of the investment universe consisting of the M risky assets (and of course the money market account) when drift effects are taken into account. The market price of risk, i.e. the return expected for incurring risk, should of course be as large as possible. Thus, the optimal portfolio or "Market Portfolio" consisting of the M risky assets has to be constructed in such a way, that its Deutsch Ratio™ is maximal. We can summarize these insights in the following theorem:

The *market portfolio* is the *fully invested* portfolio with the maximal Deutsch Ratio™ attainable within the available investment universe. The maximum Deutsch Ratio™ is the slope of the capital market line describing the expected return of the optimal investment strategy as a function of the investment risk. Therefore, the maximum Deutsch Ratio™ is the market price of risk for the investment universe under consideration.

6.2 Deutsch Ratio™, Sharpe Ratio and Risk Adjusted Performance

We now compare the Deutsch Ratio™ defined in Eq. 4.6 with the traditional Sharpe Ratio from Eq. 2.3 everybody is accustomed to since 50 years. The reciprocal of Eq.4.6 directly yields the relation between the Deutsch Ratio™ and the Sharpe Ratio

$$\Gamma_V^{-1} = q\,\gamma_V^{-1} - 1 \Longleftrightarrow \Gamma_V = \frac{\gamma_V}{q - \gamma_V} \quad \text{with} \quad \gamma_V \equiv \frac{R_V - r_f}{\sigma_V} \tag{6.1}$$

This again explicitly shows that the Deutsch Ratio™ is well behaved ($-\infty < \Gamma_V < \infty$) only for $q > \gamma_V$ as already seen in Eq. 5.6.

Note that the Sharpe Ratio is not directly a risk adjusted performance measure ("RAPM") in the sense of expected (excess) return over the holding period δt per risk over that holding period, not even if the the drift is neglected in Eq. 3.2: The expected return over the holding period is $R_V \delta t$ and the risk free proceeds are $r_f \delta t$. If the risk (in percent of the portfolio value) is given by the VaR as in Eq. 3.2 with no drift, a risk adjusted performance measure would be

$$\begin{aligned}
\text{RAPM}\,(c, \delta t)_{\text{drift neglected}} &= \frac{R_V \delta t - r_f \delta t}{\text{VaR}_V(c)/V} \\
&\approx \frac{R_V \delta t - r_f \delta t}{|Q_{1-c}|\,\sqrt{\delta t}\,\sigma_V} \\
&= \frac{\sqrt{\delta t}}{|Q_{1-c}|}\,\frac{R_V - r_f}{\sigma_V} \\
&= \frac{1}{q}\gamma_V
\end{aligned}$$

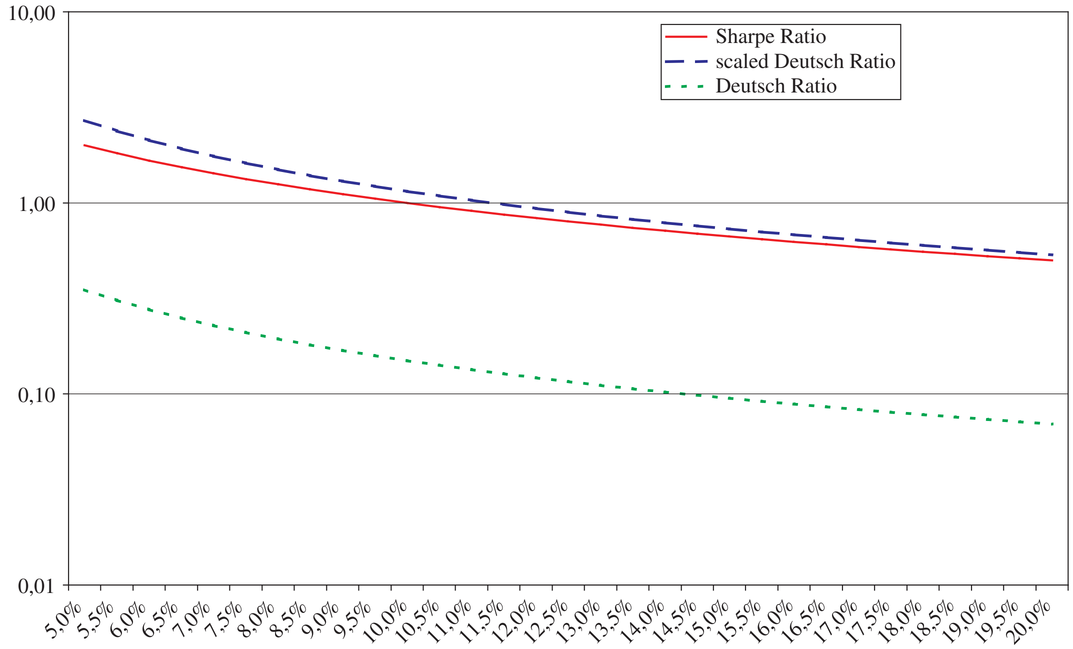

Figure 6.1: Comparison of Sharpe and Deutsch Ratio™ for 90% confidence of a portfolio with a holding period of 10 days as a function of the portfolio's volatility for a constant expected excess return of 10% (note the logarithmic scale).

with the abbreviation q as in Eq. 3.5. Thus, the Sharpe Ratio has to be "scaled" with the factor $1/q = \sqrt{\delta t}/\left|Q_{1-c}\right|$ to qualify for a (dimensionless) risk adjusted performance measure.

The Deutsch Ratio™, on the other hand, is directly a RAPM: The expected return over the holding period is again $R_V \delta t$ and the risk free proceeds are again $r_f \delta t$. But the risk (potential loss for a given confidence c in percent of the portfolio value) is now $\eta_V \delta t$ as can directly be seen from the first line of Eq. 3.4. Thus all factors δt cancel:

$$\begin{aligned} \text{RAPM}\,(c,\delta t)_{\text{risk as Eq.3.4}} &= \frac{R_V \delta t - r_f \delta t}{\text{VaR}_V(c)/V + r_f \delta t} \\ &\approx \frac{R_V \delta t - r_f \delta t}{\eta_V \delta t + r_f \delta t} \\ &= \Gamma_V \end{aligned}$$

When comparing the Deutsch Ratio™ with the Sharpe Ratio numerically, one should therefore either compare the two resulting RAPMs, or one should compare γ_V with the scaled Deutsch Ratio™ $q\Gamma_V$. Figure 6.1 shows such a comparison. From Eq. 6.1 it is easy to see, that the scaled Deutsch Ratio™ can be written as

$$q\Gamma_V = \frac{1}{\gamma_V^{-1} - q^{-1}}$$

Thus, the larger q is compared to the Sharpe Ratio, the closer (apart from the scaling) the Sharpe Ratio and the Deutsch Ratio™ become.

7 The Market Portfolio with Drift

In Section 4 we assumed that there exists an optimal portfolio which was the basis for the Capital Market Line. We will now *construct* this optimal portfolio, i.e. we will explicitly determine its weights. To do this, we search for any portfolio with maximum Deutsch Ratio, and than, in a second step scale its weights such that it is fully invested.

Let's start by finding the so-called *characteristic portfolio* (Deutsch (2004)) with respect to the numerator of our RAPM, i.e. with respect to the excess return. Thus, we take the vector $\widehat{\mathbf{R}}$ of the asset's excess returns as the so-called *attribute vector*. Then the portfolio's excess return $\widehat{R}_V = \mathbf{w}^T \widehat{\mathbf{R}}$ is the exposure of portfolio V to that attribute (Deutsch (2004)). By definition, the characteristic portfolio V_R for attribute $\widehat{\mathbf{R}}$ is the minimum risk portfolio with exposure $\widehat{R}_V = 1$. All of this is exactly the same as in Markowitz theory. But instead of the volatility we now have $\widehat{\eta}$ from Eq. 5.2 as our risk measure. Thus, the characteristic portfolio V_R for attribute $\widehat{\mathbf{R}}$ has to minimize the risk

$$\widehat{\eta}_R = q \underbrace{\sqrt{\mathbf{w}_R^T \mathbf{C} \mathbf{w}_R}}_{\sigma_R} - \underbrace{\mathbf{w}_R^T \widehat{\mathbf{R}}}_{\widehat{R}_R} \tag{7.1}$$

under the constraint

$$\mathbf{w}_R^T \widehat{\mathbf{R}} \stackrel{!}{=} 1 \tag{7.2}$$

In other words, for a *constant* portfolio excess return $\widehat{R} = 1$, we construct the portfolio with minimum risk $\widehat{\eta}$ or equivalently with maximum Ratio $1/\widehat{\eta}_V$. But since $\widehat{R}_V = 1$ by construction, $1/\widehat{\eta}_V$ is the Deutsch Ratio™ $\widehat{R}_V/\widehat{\eta}_V$. Therefore, this portfolio will have maximum Deutsch Ratio™.

The Lagrangian for this optimization problem is

$$\mathcal{L} = \underbrace{q\sqrt{\mathbf{w}_R^T \mathbf{C} \mathbf{w}_R} - \mathbf{w}_R^T \widehat{\mathbf{R}}}_{\text{To be Minimized}} - \lambda \underbrace{\left[\mathbf{w}_R^T \widehat{\mathbf{R}} - 1\right]}_{\text{Constraint}}$$

To find the optimal weights $\mathbf{w}_R$ we differentiate this Langrangian with respect to those weights and set the derivative equal to zero.

$$0 \stackrel{!}{=} \frac{\partial \mathcal{L}}{\partial \mathbf{w}_R^T} = q \frac{\mathbf{C} \mathbf{w}_R}{\sqrt{\mathbf{w}_R^T \mathbf{C} \mathbf{w}_R}} - (1+\lambda)\widehat{\mathbf{R}} \tag{7.3}$$

Multiplying from the left by $\mathbf{w}_R^T$ and observing Eq.7.2 yields the Lagrange multiplier.

$$
\begin{aligned}
0 &= q\frac{\mathbf{w}_R^T\mathbf{C}\mathbf{w}_R}{\sqrt{\mathbf{w}_R^T\mathbf{C}\mathbf{w}_R}} - (1+\lambda)\underbrace{\mathbf{w}_R^T\widehat{\mathbf{R}}}_{1}\\
1+\lambda &= q\sqrt{\mathbf{w}_R^T\mathbf{C}\mathbf{w}_R} = q\sigma_R
\end{aligned}
$$

With this λ, the weights in Equation 7.3 become

$$
\begin{aligned}
\mathbf{w}_R &= \frac{1}{q}\sqrt{\mathbf{w}_R^T\mathbf{C}\mathbf{w}_R}(1+\lambda)\mathbf{C}^{-1}\widehat{\mathbf{R}}\\
&= \sigma_R^2\mathbf{C}^{-1}\widehat{\mathbf{R}}
\end{aligned}
\tag{7.4}
$$

Note at this point that q has cancelled! Thus, we have the reassuring result

The weights of the optimal portfolio which maximizes the Deutsch Ratio™ are independent of an investors holding period δt and confidence level c.

To proceed further, it is easiest to left-multiply by $\widehat{\mathbf{R}}^T$, exploiting Constraint 7.2 again:

$$
\begin{aligned}
\widehat{\mathbf{R}}^T\mathbf{w}_R &= 1\\
\sigma_R^2\,\widehat{\mathbf{R}}^T\mathbf{C}^{-1}\widehat{\mathbf{R}} &= 1
\end{aligned}
$$

This directly yields the variance of the characteristic portfolio:

$$
\sigma_R^2 = \frac{1}{\widehat{\mathbf{R}}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}
$$

Thus, the weights in Eq. 7.4 are explicitly

$$
\mathbf{w}_R = \frac{\mathbf{C}^{-1}\widehat{\mathbf{R}}}{\widehat{\mathbf{R}}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}} \tag{7.5}
$$

As enforced by constraint 7.2, this portfolio has an excess return of 1, i.e., of 100%. Therefore it usually contains significant leverage. Let's now determine this leverage, i.e., the degree of investment in risky assets. The part of the investment which is invested in risky assets is

$$
w = \mathbf{1}^T\mathbf{w}_R = \frac{\mathbf{1}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}{\widehat{\mathbf{R}}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}
$$

and the weight of the cash account is accordingly $(1-w)$. *The Market Portfolio* for our risk measure η is by definition, analogous to Markowitz theory, the *fully invested* portfolio with maximum Deutsch Ratio™. To find this Market Portfolio we simply divide the weights of the above optimal portfolio by the risky portion w:

$$
\begin{aligned}
\mathbf{w}_m &= \frac{1}{w}\mathbf{w}_R\\
&= \frac{\widehat{\mathbf{R}}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}{\mathbf{1}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}\frac{\mathbf{C}^{-1}\widehat{\mathbf{R}}}{\widehat{\mathbf{R}}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}\\
&= \frac{\mathbf{C}^{-1}\widehat{\mathbf{R}}}{\mathbf{1}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}
\end{aligned}
$$

But these are exactly the same weights as the weights of the Market Portfolio in Markowitz theory, i.e. with the volatility as the risk measure, see Eq. 2.4. We can therefore state:

The portfolio which maximizes the Deutsch Ratio™ also maximizes the Sharpe Ratio.

Its expected excess return, volatility and its risk are

$$
\begin{aligned}
\widehat{R}_m &= \widehat{\mathbf{R}}^T\mathbf{w}_m = \frac{\widehat{\mathbf{R}}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}{\mathbf{1}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}\\
\sigma_m^2 &= \mathbf{w}_m^T\mathbf{C}\mathbf{w}_m = \frac{\widehat{\mathbf{R}}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}{\left(\mathbf{1}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}\right)^2} = \frac{\widehat{R}_m}{\mathbf{1}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}\\
\widehat{\eta}_m &= q\sigma_m - \widehat{R}_m = \frac{\sqrt{\widehat{\mathbf{R}}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}}{\mathbf{1}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}\left(q - \sqrt{\widehat{\mathbf{R}}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}\right)
\end{aligned}
\tag{7.6}
$$

8 Summary

In this paper we have introduced the *Deutsch Ratio*™ which is the correct market price of risk when drift effects are taken into account. This ratio (and not the Sharpe Ratio) emerges naturally when *excess* returns (instead of returns) are considered throughout. The *Capital Market Line* describing the expected excess return of the optimal investment strategy as a function of the risk incurred is given by Eq.5.5, i.e.,

$$
\widehat{R}_V = \Gamma_m\widehat{\eta}_V \quad \text{with} \quad \Gamma_m = \frac{\widehat{R}_m}{\widehat{\eta}_m}
$$

This is the central result of this paper. We have also shown by explicit construction that the *Market Portfolio* defining this capital market line is the same as in traditional Markowitz theory. Therefore, even when drift effects are taken into account there still exists *the* Market Portfolio everybody should invest (part of his/her money) in. And portfolio optimization is as stable and parameter-independent (w.r.t. holding period and confidence) when maximizing the Deutsch Ratio™ as it is when maximizing the Sharpe Ratio, as long as holding period and confidence are chosen in a sensible way, i.e., as long as they fulfill Eq.5.6.

Although the Market Portfolio is the same as the Markowitz Market Portfolio and therefore independent of holding period and confidence, any individual portfolio within the optimal strategy for a specific risk preference η_V is still dependent on δt and c, since η_V depends on those parameters. The optimal extent of investment w is determined by the risk preference η_V via Eq. 4.3, i.e.,

$$
w = \frac{\widehat{\eta}_V}{\widehat{\eta}_m}
$$

with $\widehat{\eta}_m$ from Eq. 7.6. Therefore, although the portfolio to construct the Capital Market Line is the same as in Markowitz theory, any portfolio on that line differs from Markowitz theory. This is because any portfolio on that line has to fulfill a constraint regarding the preferred *risk* (which is different from the Markowitz volatility) while the Market Portfolio itself only has to fulfill the "fully invested" constraint which is the same as in Markowitz theory.

All these concepts (among many others) have been implemented in d-fine's *triple-α* portfolio optimization service[10]. This service is also capable of using much more realistic risk measures like model- and parameter-*free* Value at Risk and Expected Short Fall methods, taking full account of fat tail information and extreme events like market crashes. To be fully realistic, we also consistently include transaction costs, borrowing fees and mandate compliance constraints in the optimization decisions. Together with other concepts like characteristic portfolios and stop loss strategies based on GARCH volatility models, superb *and* realistic performance can be generated. Figure 8.1 shows a typical example for such a portfolio performance over almost 12 years (from June 1992 until March 2004).

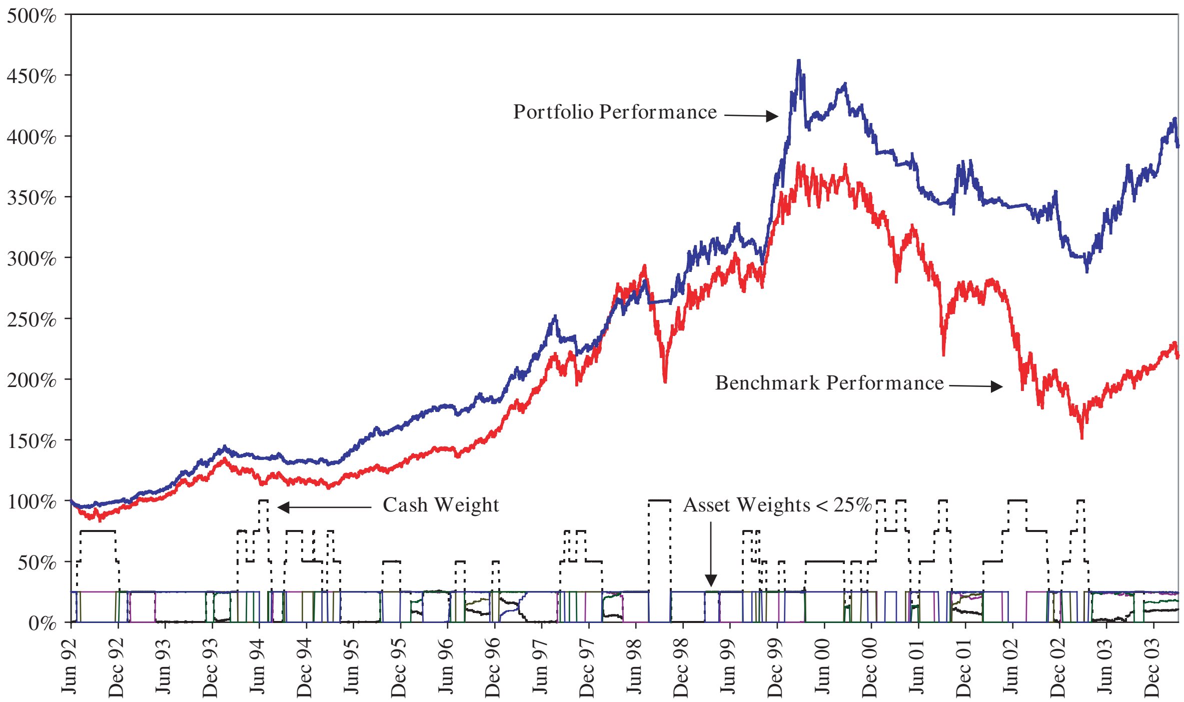

Figure 8.1: A typical portfolio performance over 12 years. The risky assets are ETFs on three DJ STOXX 600 Sector Indexes (Media, Healthcare and Technology) and an ETF on the MDAX. The Benchmark is the DJ STOXX 600 itself. The Deutsch Ratio™ was optimized under the constraints that no short selling was allowed and that the weight of each risky asset has to always be ≤25%. The risk was kept as close to the risk of the benchmark as possible within these constraints (but always lower than the benchmark risk). In addition each position was stop loss managed based on its current volatility. Transaction costs were fully taken into account. They are the reason for the relatively low trading frequency despite the fact that the optimizer was "allowed" to trade every single trading day. The annualized average Alpha over those 12 years is 6.93%.

FOOTNOTES & REFERENCES

1. In this paper we always use "hats" to denote *excess* returns.
2. Since $\delta \ln(V) = \ln V(t+\delta t) - \ln V(t+\delta t) = \ln\left[\frac{V(t+\delta t)}{V(t)}\right]$ and by the very definition of a return we have $V(t+\delta t) = V(t)e^{r_V(t)\delta t}$, i.e., $\delta \ln(V) = r_V(t)\delta t$.
3. Note, however, that the r_V are all annualized.
4. Similarly, the investment's expected *excess* returns is simply

$$\widehat{R}_V = w\widehat{R}_m$$

5. Deutsch Ratio™ is a Trademark of **d-fine** GmbH, Frankfurt, Germany.
6. For all investments on the Capital Market Line we of course have $\Gamma_V = \Gamma_m$.
7. This is the fundamental difference to a risk measure based solely on the volatility like, e.g., Eq.2.2. If such a risk measure is zero then there are no fluctuations at all by definition.
8. For instance for a confidence of $c = 50\%$ the percentile of the standard normal distribution is $Q_{1-c} = 0$ and the "risks" are always $\widehat{\eta}_V = -\widehat{R}_V$ and $\eta_V = -R_V$ no matter how large the fluctuations are! This clearly shows that some parameter choices are utterly senseless!
9. This also holds if one prefers η_V as the risk measure, see Eq. 4.4.
10. See *http://www.d-fine.de/triple_a*

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