The Martingale Optimality Principle: The best you can is good enough

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1 Introduction

The word martingale is definitely among the most used words in mathematical finance, a fact which is due to the fundamental importance of martingale measures in connection with option pricing. However, this subject will not be touched here. The area of application where this article is centered around is not pricing but optimal behaviour of an individual at a financial market or at any area where decisions about control actions have to be taken such as looking for optimal investment strategies, steering an airplane in an efficient way, or searching for the optimal velocity of a production line. What makes the appearance of a martingale mysterious in this connection is its usual interpretation as an equilibrium or as a model for a fair game where one is (in the mean) as rich after having played the game as one was before participating in it. And this is definitely not the aim of someone optimizing his income !

Tomorrow will be like today

So let us first define what a martingale is:

Definition 1:
A ***martingale*** $\{(X(t), F_t), t \in [0, T]\}$ is a (real-valued) integrable stochastic process satisfying

$$E(X(s)|F_t) = X(t) \text{ a.s. for all } s, t \in [0, T] \text{ with } s \geq t$$

where we also allow the fixed constant T to equal plus infinity.

In this sense, a martingale can be characterized by the phrase "Tomorrow will be like today". However, this phrase leaves a lot of space for different mathematical consequences. Typical examples are given by the pictures below all displaying realizations of a martingale and showing a fair random walk with constant step size (i.e. the probability for an up move of 1 equals that of a down move of 1), a fair random walk with increasing step size (i.e. at time $t = n$ the step size equals $\pm n$), and a fair random walk with decreasing step size $1/n$.

Still, the main question remains: Why should such a behaviour of your wealth or of your optimal utility or of any other thing that you can control should be desirable ? Or:

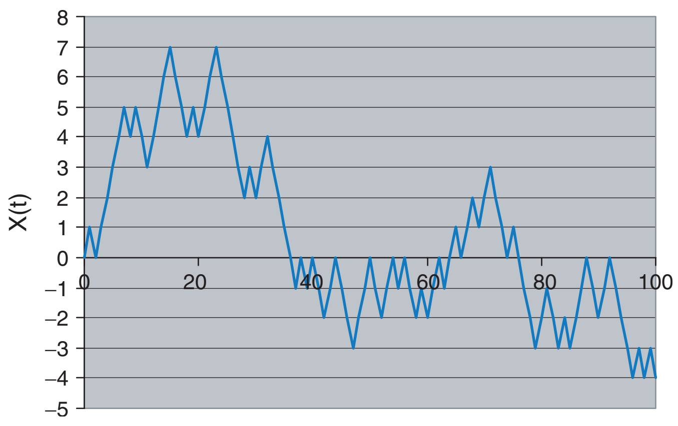

Figure 1: A fair random walk

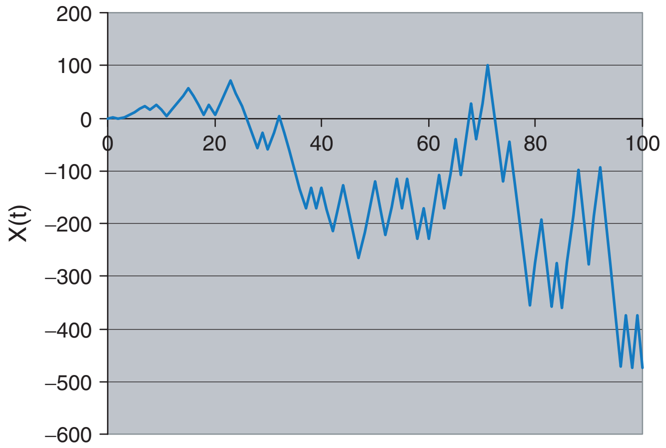

Figure 2: A fair random walk with increasing step size

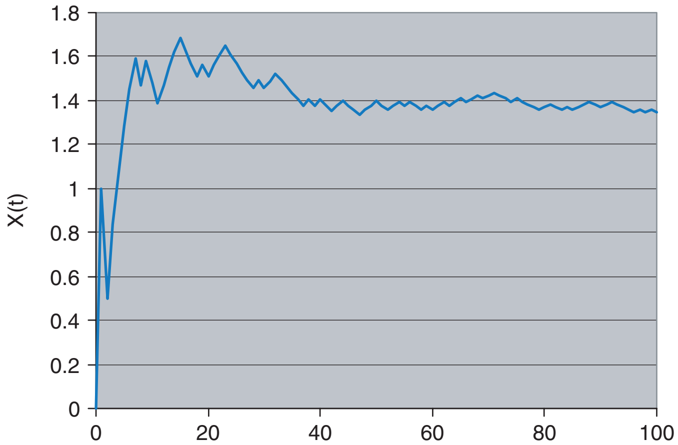

Figure 3: A fair random walk with decreasing step size

When should you hope that tomorrow is like today ?

The general answer is simple. If today is already brilliant then it would be nice if tomorrow would be the same. A very intuitive example from sports is given by your football team being on top of the table. It simply cannot get better ! Thus, it would be optimal if one could conserve this state over time. The mathematical answer to the question will be given by the martingale optimality principle. It is a general principle which can on one hand be used to solve optimal control problems and on the other hand can be used to derive nearly all optimality characterizations in the area of stochastic control independent on the dynamics of the underlying controlled process. However, its proof is extremely simple. We will state it in a very general form and demonstrate how optimality and existence proofs of control strategies can be separated by using it. Special applications are intuitive derivations of the HJB-equation and the quasi-variational inequalities of impulse control. The direct applications in finance will be the solution of both the standard Merton problem of portfolio optimization and a verification theorem for the portfolio problem under fixed and proportional transaction costs.

2 The Martingale Optimality Principle

Let $X^u(t)$ be a controlled stochastic process, i.e. a stochastic process depending on a strategy $u(.)$ ("the control") which we are allowed to choose. Here, the time set can be either continuous (in which case we concentrate on the intervall $[0, T]$) or discrete (in which case we consider the finite set $\{0, 1, \ldots, T\}$), the state space should be some subset of IR^d and the control $u(.)$ should satisfy conditions such that the resulting controlled process exists and is uniquely determined by the choice of the control. Of course, the control should only be chosen with respect to the knowledge of the past and present of the controlled process, i.e. it should be progressively measurable with respect to a suitable filtration. We further assume the controlled process to be a Markov process. The technical assumptions will be made more precise when we give various concrete applications of the martingale optimality principle in the next section.

Our aim is the solution of the following control problem

$$\max_{u(.)\in A(x,I)} E^{0,x}(F(X^u(T))), \tag{1}$$

i.e. we maximize the expected utility of the final value of the controlled process where suitable smoothness assumptions on the utility function F are specified when needed and where $A(x, I)$ denotes the set of admissible control processes on the time set I given that the initial state of the controlled process is x. Exact specifications of an admissible control will depend on the nature of the problem and the dynamics of the underlying process and will therefore be given in the different applications.

The principle method of stochastic control now is to look at a whole family of such control problems parametrized by all possible initial states of the time and the controlled process (t, x) which gives rise to the so called value function

$$v(t, x) = \sup_{u(.)\in A(x,I(t))} E^{t,x}(F(X^u(T))) \tag{2}$$

where $I(t)$ denotes the restricted time set starting at time t.

Having introduced this minimal set of ingredients we are now already able to formulate the martingale optimality principle.

Theorem 1: *The Martingale Optimality Principle*

If there exists a control strategy $u^*(.)$ such that with the definition of the function

$$g(t, x) := E^{t,x}\left(F\left(X^{u^*}(T)\right)\right) \tag{3}$$

we have that

(M) $g\left(t, X^{u^*}(t)\right)$ is a martingale

(SM) $g(t, X^u(t))$ is a supermartingale for all admissible control strategies $u(.)$

then we obtain:

a) $u^*(.)$ is an optimal control strategy for problem (1).

b) For all possible initial states of the time and the controlled process (t, x) we get

$$g(t, x) = v(t, x), \tag{4}$$

i.e. $g(.,.)$ coincides with the value function of problem (1).

Proof: We first prove assertion a). Let therefore $u(.)$ be an arbitrary admissible control and $u^*(.)$ an admissible control such that the corresponding function $g(.,.)$ as defined in (3) satisfies the conditions (M) and (SM). We then obtain

$$E^{0,x}\left(F\left(X^{u^*}(T)\right)\right) = E^{0,x}\left(g\left(T, X^{u^*}(T)\right)\right) = g(0, x) \geq E^{0,x}(g(T, X^u(T)))$$
$$= E^{0,x}(F(X^u(T))),$$

where we have used the properties (M) and (SM). Hence, optimality of $u^*(.)$ is proved. To obtain assertion b) just replace the pair $(0, x)$ by (t, x)in the above chain of equalities and inequalities (where we of course implictly use the Markov property of the controlled processes).

Remark: The simplicity of the proof of the martingale optimality principle is a bit surprising given the usually long and technical proofs of optimality results (compare e.g. Fleming and Soner (1993) or Korn and Korn (2001)). However, it will turn out that the existence of a control process with the properties (M) and (SM) is quite a strong assumption, but it will also imply all the popular optimality results of stochastic control. Of course, besides existence, the main question is how to get such a strategy. We will deal with this problem in the following explicit examples in the next section.

3 The Martingale Optimality Principle in Control and in Finance

i) ***The HJB-Equation of stochastic control and Merton's portfolio problem***

We first consider the classical case of a controlled stochastic differential equation, i.e. we assume that the controlled process $X^u(t)$ is the solution of a stochastic differential equation

$$dX^u(t) = \mu(t, X^u(t), u(t))dt + \sigma(t, X^u(t), u(t))dW(t) \tag{5}$$

where the time set is the intervall $[0, T]$, $W(t)$ a one-dimensional Brownian motion, and the coefficient functions are uniformly (with respect to time t and the control u) Lipschitz in the x-variable. We consider problem (2) and assume that F is continuous, concave and grows at most polynomially in x.

To make use of the martingale optimality principle we consider some arbitrary control process $u(.)$ and assume that the function

$$h(t, x) := E^{t,x}(F(X^u(T))) \tag{6}$$

is a $C^{1,2}$-function. Application of Itô's formula together with representation (5) leads to

$$\begin{aligned} h(t, X^u(t)) = h(0, x) &+ \int_0^t h_x(s, X^u(s))\sigma(s, X^u(s), u(s))dW(s) \\ &+ \int_0^t [h_t(s, X^u(s)) + h_x(s, X^u(s))\mu(s, X^u(s), u(s)) \\ &+ \tfrac{1}{2}h_{xx}(s, X^u(s))\sigma^2(s, X^u(s), u(s))]ds. \end{aligned} \tag{7}$$

Under suitable growth conditions (such as $h(.)$ being polynomially bounded) the stochastic integral in (7) is a martingale. Hence, $h(t, X^u(t))$ would be a martingale if the integrand in the ds-integral would vanish. It would be a supermartingale if this integrand would always be non-positive. Taking into account the martingale optimality principle, a control process with an associate function $h(.,.)$ as given in (6) would be optimal, and further $h(.,.)$ would coincide with the value function. We have now totally specified all our requirements on both the optimal control $u(t)$ and the corresponding function $h(.,.)$ and by using the martingale optimality principle we have thus proved the following well-known result:

Theorem 2: *Verification theorem for the HJB-Equation*

Let there exist a polynomially bounded classical $C^{1,2}$-solution to the **Hamilton-Jacobi-Bellman Equation** of stochastic control

$$0 = \sup_{u \in U}\{g_t(t, x) + \mu(t, x, u)g_x(t, x) + \tfrac{1}{2}\sigma^2(t, x, u)g_{xx}(t, x)\} \tag{8}$$

$$F(x) = v(T, x) \tag{9}$$

Then with $v(t, x)$ denoting the value function of problem (2) we have:

a) $g(t, x) \geq v(t, x)\ \forall\,(t, x) \in [0, T) \times \mathbf{R}$.

b) If there exists an admissible control $u^*(t)$ satisfying

$$\begin{aligned} u^*(t) \in \underset{u \in U}{arg\,max} &\left\{g_t\left(t, X^{u^*}(t)\right) + \mu\left(t, X^{u^*}(t), u\right) g_x\left(t, X^{u^*}(t)\right)\right. \\ &\left. + \tfrac{1}{2}\sigma^2\left(t, X^{u^*}(t), u\right) g_{xx}\left(t, X^{u^*}(t)\right)\right\} \end{aligned}$$

then we have

$$g(t, x) = v(t, x)\ \forall\,(t, x) \in [0, T) \times \mathbf{R}, \tag{10}$$

and $u^*(t)$ is an optimal control strategy for problem (2).

Remark:

1. It is worth pointing out that now the proofs of optimality and existence for an optimal control are well separated. While the martingale

optimality principle ensures the optimality, the existence of such a control is implied by the existence of a sufficiently smooth solution of the HJB-Equation.

2. Of course, the relevant variants for verification results for HJB-Equations for problems with running costs, with constrained state space or with an infinite horizon (see Fleming and Rishel (1975) or Fleming and Soner (1993) for an overview on the different forms of the continuous-time stochastic control problems) can be derived in a similar fashion.

Example: Merton's optimal portfolio problem

An even more explicit example than the above one of Theorem 2 is given by Merton's portfolio problem (see Merton (1969), (1971), (1990)). We consider the special case of a terminal wealth problem: Given that an investor invests in a riskless bank account and a risky stock with price dynamics given by

$$\begin{aligned} dP_0(t) &= P_0(t)rdt, P_0(0) = 1, \\ dP_1(t) &= P_1(t)[bdt + \sigma dW(t)], P_1(0) = p \end{aligned} \tag{11}$$

(where $W(t)$ is a one-dimensional Brownian motion) the investor's goal is to maximize

$$\max_{\pi(.)\in A(x)} E^{0,x}(U(X^{\pi}(T))) \tag{12}$$

where the utility function $U(x)$ is assumed to be strictly increasing and strictly concave (typical examples are $\ln(x)$ or x^{γ} with $0 < \gamma < 1$). Here, $\pi(.)$ denotes the investor's portfolio process (i.e. the process of the fraction of his wealth invested in the stock at time t) and $A(x)$ is the set of all admissible portfolio processes if the investor has an initial endowment of x (see Korn and Korn (2001) for more details). For simplicity, we will only call a bounded stochastic process which is progressively measurable with respect to the Brownian filtration an admissible portfolio process. Such a portfolio process in particular leads to a non-negative wealth process which can be seen from the following stochastic differential equation for the evolution of the investor's wealth $X(t)$ over time:

$$dX(t) = X(t)[(r + \pi(t)(b - r))dt + \pi(t)\sigma dW(t)], X(0) = x \tag{13}$$

(see e.g. Chapter 2 in Korn and Korn (2001) for the derivation of this equation). Merton's ingenious idea then was to interpret this wealth equation (13) as a controlled stochastic differential equation of the form (5) where the control strategy is the portfolio process. I.e. we obtain such an equation by setting

$$u(t) = \pi(t), \quad \mu(t, x, u) = (r + u(b - r))x, \quad \sigma(t, x, u) = u\sigma x. \tag{14}$$

By Theorem 2, we can solve the portfolio problem (12) via solving the corresponding HJB-Equation

$$\begin{aligned} 0 = \max_{\pi\in[-a,a]} &\{\tfrac{1}{2}(\sigma\pi x)^2 v_{xx}(t, x) + (r + \pi(b - r)) \\ &\times x v_x(t, x) + v_t(t, x)\}, U(x) = v(T, x) \end{aligned} \tag{15}$$

for the value function

$$v(t, x) = \max_{\pi} E^{t,x}(U(X(T))) \,. \tag{16}$$

In the case of a so-called HARA-utility function $U(x) = \frac{1}{\gamma}x^{\gamma}$ with $0 < \gamma < 1$ (or in case of the logarithm $U(x) = ln(x)$) one assumes the value function to be concave. With this assumption the optimization problem in (15) consists of maximizing a parabola which is downwards open. Thus, the formal minimizer in (15) is given by

$$\pi^*(t) = \pi^*(t, x) = -\frac{(b - r)v_x(t, x)}{\sigma^2 x v_{xx}(t, x)}. \tag{17}$$

Plugging this into (15) results in a non-linear partial differential equation which can be solved explicitly using the ansatz $v(t, x) = f(t)x^{\gamma}$ (see Korn and Korn (2001)). Having verified this, we then obtain the optimal portfolio process from (17) as

$$\pi^*(t) = \frac{b - r}{(1 - \gamma)\sigma^2} \tag{18}$$

(the optimal solution for the logarithmic utility is obtained by setting $\gamma = 0$). For more on the topic of the Merton problem and generalizations (multi-stock setting, incomplete markets, constraints, ...) see e.g. Karatzas and Shreve (1998) or Korn (1997). Note that the constraint $\pi \in [-a, a]$ implicitly introduced in (15) is only needed to ensure a global Lipschitz condition for the wealth equation. The constant a is at our disposal and we just have to choose it large enough such that the optimal portfolio process given by (18) lies in the interior of $[-a, a]$.

ii) *Portfolio optimization with transaction costs and the quasi-variational inequalities of of impulse control*

The above optimal portfolio process given in (18) is constant over time which is a nice structural feature. However, from the application's point of view this means that an investor has to trade at each point in time to keep the fraction of the wealth invested in the stock constant as bond and stock prices change in different ways. So in the presence of fixed transaction costs the investor would immediately (!) be ruined. To overcome this problem, Eastham and Hastings (1988) introduced an impulse control approach to portfolio optimization which was later taken up and generalized in Korn (1998).

We consider the same securities market as given in equation (11) of the previous section. Now the trading strategy of an investor is completely described by the money invested in bond and stock at time t, $(B(t), S(t))$. As long as he does not rebalance these holdings they evolve as multiples of the relevant security prices, i.e. we have

$$\begin{aligned} dB(t) &= B(t)r\,dt, B(0) = B, \\ dS(t) &= S(t)[bdt + \sigma dW(t)], S(0) = S \end{aligned} \tag{19}$$

If however the investor decides to rebalance his holdings at an *intervention time* θ_i (i.e. at such a time where rebalancing of the holdings takes place) the investor has to pay a sum of fixed and proportional transaction costs of the form

$$K + k|\Delta S_i| \tag{20}$$

with $0 < K, 0 \leq k < 1, \Delta S_i := S(\theta_i) - S(\theta_{i-1})$. These transaction costs are paid out of the bond holdings. Therefore, we have the following balance equation:

$$B(\theta_i) = B(\theta_{i-1}) - \Delta S_i - k|\Delta S_i| - K. \tag{21}$$

Hence, the strategy of the investor is uniquely determined by the choice of the intervention times θ_i, $i \in \{0, 1, 2, \ldots\}$ and the corresponding changes ΔS_i in the stock holdings. A sequence of such pairs $(\theta_i, \Delta S_i)$, $i \in \{0, 1, 2, \ldots\}$ where θ_i is a stopping time with respect to the Brownian filtration f_t and where ΔS_i is $f\theta_i$—measurable is called an *impulse control strategy*. Note that in contrast to the situation in Merton's problem the choice of the right time to intervene adds a new decision variable to the problem which again consists of maximizing the expected utility from final wealth,

$$\max_{\{(\theta_i, \Delta S_i)\}_{i \in IN} \in Z} E^{0,B,S}(U(X(T))). \tag{22}$$

where the wealth process $X(t)$ is given by $X(t) = B(t) + S(t)$. The set of admissible impulse control strategies Z consists of all those impulse control strategies where the intervention times do not accumulate before the time horizon T and where the actions ΔS_i are constrained to yield non-negative values for $B(\theta_i)$ and $S(\theta_i)$. This ensures that the wealth of our holdings after all securities are sold is always bounded from below by $-K$. We could also require to have a non-negative wealth after selling all securities, but for simplicity we drop this condition here.

Again, the question is **how to use the martingale optimality principle** to solve problem (22), or more precisely, to characterize the optimal solution via a verification theorem?

Let us therefore consider an arbitrary admissible impulse control strategy $(\theta_i, \Delta S_i)$, $i \in \{0, 1, 2, \ldots\}$ and look at the evolution of its corresponding expected utility over time:

$$\begin{aligned} E^{t,B(t),S(t)}(U(X^{\theta,\Delta S}(T))) &=: v(t, B(t), S(t)) = v(0, B, S) \\ &+ \int_0^t [v_t(.) + bS(u)v_s(.) + \tfrac{1}{2}\sigma^2 S^2(u)v_{ss}(.)]ds + \int_0^t \ldots dW(u) \\ &+ \sum_{i=1}^{\infty}[v(\theta_i \wedge t, B(\theta_i - \wedge t) - K - \Delta S, S(\theta_i \wedge t) + \Delta S_i) \\ &- v(\theta_i \wedge t, B(\theta_i - \wedge t), S(\theta_i - \wedge t))] \end{aligned} \tag{23}$$

Simply for saving space we have omitted dependencies on (t, B, S). Due to the discontinuities of the controlled process of bond and stock holdings at the intervention times, we can apply the Itô formula only between the interventions and have to consider the difference in the values of $v(.,.,.)$ before and after interventions separately. But if we now have a close look at equation (23), we can exactly figure out the necessary conditions needed to apply the martingale optimality principle.

All expressions in (23) in the brackets have to equal zero for the left hand side being a martingale. They have to be non-positive if the left hand side should be a super martingale. To formulate this in a compact fashion we have to introduce the following two operators:

$$Mv(t, B, S) := \max_{\Delta S \in A(B,S)} v(t, B - K - \Delta S - k|\Delta S|, S + \Delta S) \tag{24}$$

where $A(B, S)$ is the feasible set for the actions given the holdings of (B,S). Note that the transaction costs directly enter into the components of v but *not* as an additional term in the maximisation problem as would be typical for impulse control problems. We further introduce:

$$Lv(t, B, S) := \tfrac{1}{2}\sigma^2 S^2\, v_{SS}(t, B, S) + bSv_S(t, B, S) + v_t(t, B, S). \tag{25}$$

To ensure that each term of the series in equation (23) is non-positive we have to require

$$v(t, B, S) \geq Mv(t, B, S), \tag{26}$$

a fact which—if $v(t,B,S)$ is the value function—is also very natural as it says: "*Behaving optimally on a global scale is always as good as doing the best immediate transaction an behaving optimally afterwards*". To ensure non-positivity of the *ds*-integrand in (23) we need

$$Lv(t, B, S) \leq 0. \tag{27}$$

Of course, these two inequalities have to be connected in some way. If the investor decides to not change his holdings then the wealth process evolves as a diffusion with characteristic operator L. During these times we must have equality in (27). However, at the first time when changing the holdings is optimal, we must have equality in (26), simply by definition of the operator M and the fact that v should be the value function. We thus get the following equality closing the system (26) and (27):

$$Lv(t, B, S)(v(t, B, S) - Mv(t, B, S)) = 0. \tag{28}$$

Together with the obvious final condition

$$v(T, B, S) = U(B + S) \tag{29}$$

(where for simplicity we have assumed no selling costs at the terminal time or—equivalently—have assumed to maximize paper value of our holdings) we have now specified all the conditions characterizing the value function, the so called *quasi-variational inequalities*, a result summarized in:

Theorem 3: *Verification theorem for the quasi-variational inequalities*
Let there exist a polynomially bounded $C^{1,2}$-solution $v(t, b, s)$ to the **quasi-variational inequalities** of impulse control (26)-(29).

If then $\hat{v}(t, b, s)$ denotes the value function of problem (22) then we have:
a) $v(t, b, s) \geq \hat{v}(t, b, s)\ \forall\,(t, b, s) \in [0, T) \times [0, \infty)^2$.
b) If there exists an admissible impulse control $\{(\theta_i, \Delta S_i)\}_{i \in N}$ given by

$$\theta_0 := 0, \theta_i := \inf\{t < \theta_{i-1} | v(t, B, S) = Mv(t, B, S)\},$$
$$\Delta S_i := \underset{\Delta S}{arg\ max}\{v(t, B - K - \Delta S - k|\Delta S|, S + \Delta S)\}$$

(a so called qvi-control) then we have

$$v(t, b, s) = \hat{v}(t, b, s)\ \forall\,(t, b, s) \in [0, T) \times [0, \infty)^2, \tag{30}$$

and the above impulse control is an optimal control strategy for problem (22).

Remark: The above result is of the form "A sufficiently regular solution of the qvi coincides with the value function and the corresponding qvi-control is an optimal one". However, this has to be handled with a lot of care. There will typically not exist a solution to the qvi satisfying the above smoothness requirements. However, it can be shown that this requirement can be relaxed to cover practical cases (see Korn (1998) for details). The main difficulty now lies in the task of *solving* the qvi. An explicit solution does not seem to be possible. However, one can try numerical methods such as discretisation of the qvi or an asymptotic expansion as given in Korn (1998) for the case of the utility function $U(x) = 1 - e^{-\lambda x}$ for some positive constant λ.

iii) *Optimality equations in discrete time*
If we are in a discrete-time setting then we can of course not hope for a differential equation characterizing the value function. However, we can then deduce the well-known Bellman or backward induction principle from the martingale optimality principle. To see this, in a discrete-time setting, let

$$h(t, x) := E^{t,x}(F(X^u(T))) \tag{31}$$

for some admissible control u (where we assume that admissibility of the control means that it is adapted to a specified filtration, it takes on only values in a compact set U, and that the controlled process satisfies integrability constraints ensuring that $h(t, x)$ is finite). It is then clear that we have

$$h(T, x) = F(x), h(T-1, x) = E^{T-1,x}(F(X^u(T))) = E^{T-1,x}(h(T, X^u(T))). \tag{32}$$

Using the tower law of conditional expectation and an induction argument we can use the last equality to show

$$h(T-k, x) = E^{T-k,x}(h(T-k+1, X^u(T-k+1))). \tag{33}$$

Now this equation together with the tower law implies that

$$h(t, X^u(t)) = E^{t,X^u(t)}(F(X^u(T))) \tag{34}$$

is a martingale. For any other admissible control w the supermartingale property of

$$h(t, X^w(t)) = E^{t,X^w(t)}(F(X^u(T))) \tag{35}$$

would be implied by the inequality

$$h(t, X^w(t)) \geq E^{t,X^w(t)}(h(t+1, X^w(t+1))). \tag{36}$$

But this inequality will always be correct if in each possible pair (t, x) we have that the control $u(t) = u(t, x)$ satisfies

$$u(t, x) = \underset{w \in U}{arg\ max}\, E^{t,x}(h(t+1, X^w(t+1))) \tag{37}$$

where here $X^w(t+1)$ means that the control w is applied at time t and results in $X^w(t+1)$. Hence, the martingale optimality principle implies:

Theorem 4: *Backward induction principle*
Assume that there is a continuous function $h(t, x)$ satisfying

$$h(t, x) = \max_{w \in U} E^{t,x}(h(t+1, X^w(t+1))) \tag{38}$$
$$h(T, x) = F(x) \tag{39}$$

for all $t \in \{0, 1, .., T-1\}, x \in I\!R$. If further the corresponding process $u(t) = u(t, x)$ given by (37) is an admissible control then we have

$$v(t, x) = h(t, x) \tag{40}$$

for all above pairs (t, x), where $v(t, x)$ denotes the value function of problem (2), and the control $u(t) = u(t, x)$ is an optimal control for problem (2).

We thus can compute the optimal utility and the optimal control backwards starting at the final time where the value function and the utility function coincide. A possible area of application of this result is the solution of a portfolio problem in an *n*-period binomial model via the above indicated backward induction algorithm. We do not present it here but encourage the interested reader to try to solve it along the lines indicated above.

4 More Problems and More Martingales

Of course there are many more control problems in finance and in other applications than the ones presented here. However, you are now equipped with a tool box that you can successfully use to solve such problems !

Discover your own martingale

One way to apply the martingale optimality principle then is to simply guess the optimal strategy. What then remains is to show the properties (M) and (SM). A good way in guessing an optimal strategy is to look for a suitable martingale in your control problem. The typically harder part then is to show property (SM) (of course only, if your guess was correct !). The other way to apply the martingale optimality principle is the systematic construction of sufficient conditions characterizing the value function and optimal controls. This can typically be done similar to the derivation of the two verification theorems in Section 3.

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