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The Capital Growth Theory of Investment: Part II

Using the Kelly criterion for betting on favorable (unpopular) numbers in lotto games – even with a substantial edge and very large payoffs if we win – the bets are extremely tiny because the chance of losing most or all of our money is high.

Lotteries predate the birth of Jesus. They have been used by various organizations, governments and individuals to make enormous profits thanks to the greed and hopes of lottery players who wish to turn tens into millions. The Sistine Chapel in the Vatican, including Michelangelo's ceiling, was partially funded from lotteries. So was the British Museum. Major Ivy League universities in the US such as Harvard used lotteries to fund themselves in their early years. Former US president Thomas Jefferson used a lottery to pay off his debts when he was 83. Abuses occur from time to time and government control is typically the norm. Lotteries were banned in the US for over a hundred years and resurfaced in 1964. In the UK, the dark period was 1826-1994. Since then there has been enormous growth in lottery games in the US, Canada, the UK and other countries. Current lottery sales in the UK are about £5 billion per year. Sales of the main 6/49 lotto game average about £80 million a week. The lottery operator (Camelot) takes about 5 per cent of lotto sales for its remuneration, 5 per cent goes to retailers, 12 per cent goes to the government in taxes, and another 28 per cent goes to various good causes, as do unclaimed prizes.

One might conclude that the expected payback to the lotto player is 50 per cent of his or her stake. However, the regulations allow a further 5 per cent of regular sales to be diverted to a Super Draw fund. Furthermore we must allow for the probability that the jackpot is not won. Eighty of 567 jackpots to the end of May 2001 had not been won. This means that the expected

TABLE 1: TYPES OF LOTTERY GAMES

Payoff if you win		Complete Luck	Skill Involved
	Complete Luck	Scratch Lottery Games No hope whatsoever in analyzing such games. **Payoff:** Fixed payment *Impossible to beat*	Example: Pay $1 for a chance to pick all winners of football games on Saturday. From those who have all correct selections, one name is chosen at random and awarded $100,000. **Payoff:** Fixed payment *Possibly beatable*
	Skill Involved	6/49 6/48 6/44 6/39 6/36 6/40 5/40 7/53 Lotto Games have some skill elements by picking unpopular numbers. **Payoff:** Pari-mutuel *Possibly beatable*	Sports Pool Games in UK, Mexico, Australia, France, etc Legalized Sports Betting in Nevada Horseracing Blackjack **Payoff:** Varies, can be pari-mutuel or have fixed price per dollar wagered. *Definitely beatable*

payback in a regular draw is not much more than 40%. This is still enough to get people to play. With such low paybacks it is very difficult to win at these games and the chances of winning any prize at all, even the small ones, are low.

Table 1 describes the various types of lottery games in terms of the chance of winning and the payoff if you win. Lottery organizations have machines to pick the numbers that yield random number draws. Those who claim that they can predict the numbers that will occur cannot really do so. There are no such things as hot and cold numbers or numbers that are friends. Schemes to combine numbers to increase your chance of winning are mathematically fallacious. For statistical tests on these points, see my Guidebook. One possible way to beat pari-mutuel lotto games is to wager on unpopular numbers or, more precisely, unpopular combinations.[1] In lotto games players select a small set of numbers from a given list. The prizes are shared by those with the same numbers as those selected in the random drawing. The lottery organization bears no risk in a pure pari-mutuel system and takes its profits before the prizes are shared. I have studied the 6/49 game played in Canada and several other countries.[2] Combinations like 1, 2, 3, 4, 5, 6 tend to be extraordinarily popular: in most lotto games, there would be thousands of jackpot winners if this combination were drawn. Numbers ending in eight and especially nine and zero as well as high numbers (32+, the non-birthday choices) tend to be unpopular. Professor Herman Chernoff found that similar numbers were unpopular in a different lotto game in Massachusetts. The game Chernoff studied had four digit numbers from 0000 to 9999. He found advantages from many of those with 8, 9, 0 in them. Random numbers have an expected loss of about 55 per cent. However, sextuplets of unpopular numbers have an edge with expected returns exceeding their cost by about 65%. For example, the combination 10, 29, 30, 32, 39, 40 is worth about $1.507 while the combination 3, 5, 13, 15, 28, 33 of popular numbers is worth only about $0.154. Hence there is a factor of about ten between the best and worst combinations. The expected value rises and approaches $2.25 per dollar wagered when there are carryovers (that is when the jackpot is accumulating because it has not been won). Most sets of unpopular numbers are worth $2 per dollar or more when there is a large carryover. Random numbers, such as those from lucky dip and quick pick, and popular numbers are worth more with carryovers but never have an advantage. However, investors (such as Chernoff's students) may still lose because of mean reversion (the unpopular numbers tend to become less unpopular over time) and gamblers' ruin (the investor has used up his available resources before winning). These same two phenomena show up in the financial markets repeatedly.

Table 2 provides an estimate of the most unpopular numbers in Canada in 1984, 1986 and 1996. The same numbers tend to be the most unpopular over time but their advantage becomes less and less over time. Similarly, as stock market anomalies like the January effect or weekend effect have

TABLE 2: UNPOPULAR NUMBERS IN THE CANADIAN LOTTERY 6/49, 1984, 1986, AND 1996

	1984		1986		1996	
Rank	Number	% More Unpopular than Average	Number	% More Unpopular than Average	Number	% More Unpopular than Average
1	39	34.3	40	26.7	40	13.8
2	40	34.0	39	22.9	39	12.0
3	30	33.0	20	20.5	48	11.2
4	20	26.8	30	18.1	20	9.6
5	41	18.8	41	16.8	45	9.1
6	10	17.9	38	16.7	41	9.0
7	42	16.1	42	16.4	46	9.0
8	38	15.0	46	15.3	38	8.3
9	46	12.5	29	14.9	42	7.4
10	48	11.5	49	14.9	37	6.9
11	45	9.9	48	14.0	29	6.3
12	49	9.2	32	13.0	30	6.2
13	1	8.4	10	11.6	36	5.1
14			47	10.5	44	4.5
15			1	8.2	47	4.0
16			37	6.3	32	3.1
17			28	6.3	35	2.9
18			34	6.2	34	2.9
19			45	3.2	28	2.5

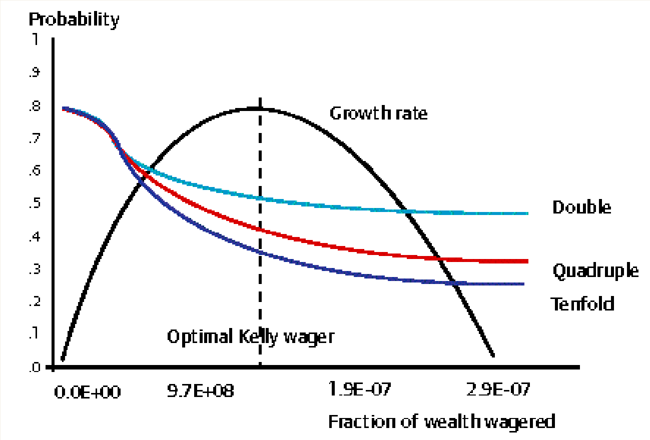
But can an investor really win with high confidence by playing these unpopular numbers? And if so, how long will it take? To investigate this, consider the following experiment detailed in Table 3.

TABLE 3: LOTTO GAME EXPERIMENTAL DATA*

Prizes	Probability of Winning	Mean Time to Win	Prize	Case A Contribution to Expected Value	Prize	Case B Contribution to Expected Value
Jackpot	1/13,983,816	134,460 y	$6 M	42.9	$10 M	71.5
Bonus, 5/6+	1/2,330,636	22,410 y	0.8 M	34.3	1.2 M	51.5
5/6	1/55,492	533 y 29 w	5,000	9.0	10,000	18.0
4/6	1/1,032	9 y 48 w	150	14.5	250	24.2
3/6	1/57	28 w	10	17.6	10	17.5
				118.1		182.7
Edge				18.1%		82.7%
Optimal Kelly Bet				0.0000011		0.0000065
Optimal Number of Tickets				11		65

Purchased per Draw with $10 M Bankroll
*Mean time in years and weeks to win if you buy one ticket in each of two draws per week
5/6+ is 5 of 6 right and the 7th number is the last one, that is 6 of 7. *Source:* MacLean and Ziemba (1999)

FIGURE 1: PROBABILITY OF DOUBLING, QUADRUPLING AND TENFOLDING BEFORE HALVING, LOTTO 6/49, CASE A



Source: *MacLean and Ziemba* (1999)

lessened over time. However, the advantages are still good enough to create a mathematical advantage in the Canadian and UK lottos.

Strategy Hint #1: when a new lotto game is offered, the best advantage is usually right at the start. This point applies to any type of bet or financial market.

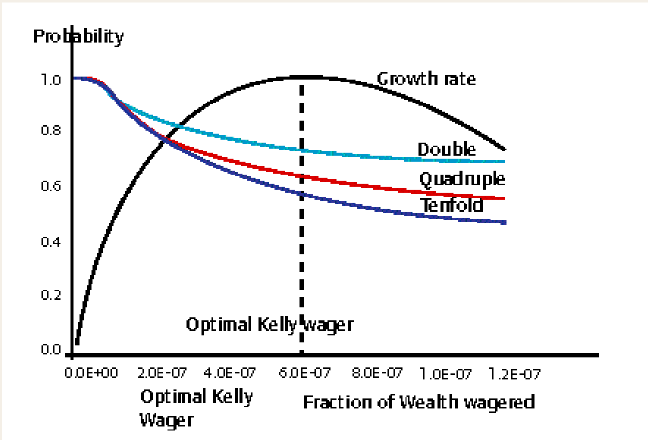
Strategy Hint #2: games with more separate events, on each of which you can have an advantage, are more easily beatable. The total advantage is the product of individual advantages. Lotto 6/49 has 6; a game with 9 is easier to beat and one with 3 harder to beat.

Case A assumes unpopular number sextuplets are chosen and there is a medium sized carryover. Case B assumes that there is a large carryover and that the numbers played are the most unpopular combinations. Carryovers (called rollovers in the UK) build up the jackpot until it is won. In Canada, carryovers build until the jackpot is won. In the UK 6/49 game, rollovers are capped at three. If there are no jackpot winners then, the jackpot funds not paid out are added to the existing fund for the second tier prize (bonus) and then shared by the various winners. In all the draws so far, the rollover has never reached this fourth rollover. Betting increases as the carryover builds since the potential jackpot rises.[3] These cases are favorable to the unpopular numbers hypothesis; among other things they correspond to the Canadian and UK games in which the winnings are paid up front (not over twenty or more years as in the US) and tax-free (unlike in the US). The combination of tax-free winnings plus being paid in cash makes the Canadian and UK prizes worth about three times those in the US. The optimal Kelly wagers are extremely small. The reason for this is that the bulk of the expected value is from prizes that occur with less than one in a million probability. A wealth level of $1 million is needed in Case A to justify even one $1 ticket. The corresponding wealth in Case B is over $150,000. Figures 1 and 2 provide the chance that the investor will double, quadruple or tenfold this fortune before it is halved using Kelly and fractional Kelly strategies for Cases A and B respectively. These chances are in the 40-60 per cent and 55-80 per cent ranges for Cases A and B, respectively. With fractional Kelly strategies in the range of 0.00000004 and 0.00000025 or less of the investor's initial wealth, the chance of increasing one's initial fortune tenfold before halving it is 95 per cent or more with Cases A and B respectively. However, it takes an average of 294 billion and 55 billion years respectively to achieve this goal assuming there are 100 draws per year as there are in the Canadian 6/49 and UK 6/49.

Figures 3 and 4 give the probability of reaching $10 million before falling to $1 million and $25,000 for various initial wealth for cases A and B, respectively, with full, half and quarter Kelly wagering strategies. The results indicate that the investor can have a 95 per cent plus probability of achieving the $10 million goal from a reasonable initial wealth level with the quarter Kelly strategy for cases A and B. Unfortunately the mean time to reach this goal is 914 million years for case A and 482 million years for case B. For case A with full Kelly it takes 22 million years on average and 384 million years with half Kelly for case A. For case B it takes 2.5 and 19.3 million years for full and half Kelly, respectively. It takes a lot less time, but still millions of years on average to merely double one's fortune: namely 2.6, 4.6 and 82.3 million years for full, half and quarter Kelly, respectively for case A and 0.792, 2.6 and 12.7 for case B. We may then conclude that millionaires can enhance their dynasties' long-run wealth provided their wagers are sufficiently small and made only when carryovers are sufficiently large (in lotto games around the world). There are quite a few that could be played.

What about a non-millionaire wishing to become one? The aspiring investor must pool funds until $150,000 is available for case B and $1 million for case A to optimally justify buying only one $1 ticket per draw. Such a tactic is legal in Canada and in fact is highly encouraged by the lottery corporation which supply legal forms for such an arrangement. Also in the UK, Camelot will supply model "agreement" forms for syndicates to use, specifying who must pay what, how much, and when, and how any prizes will be

FIGURE 2: PROBABILITY OF DOUBLING, QUADRUPLING AND TENFOLDING BEFORE HALVING, LOTTO 6/49, CASE B



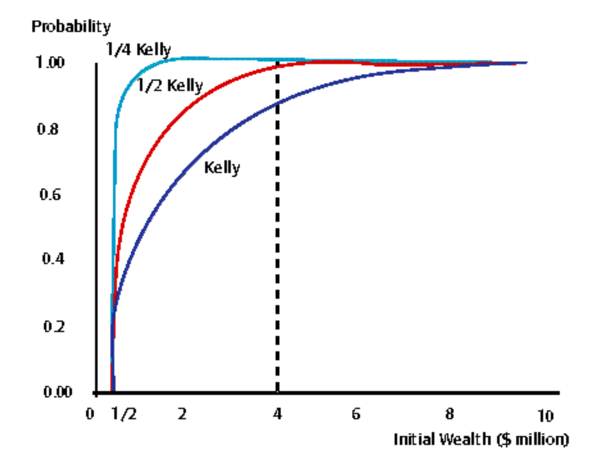
Source: *MacLean and Ziemba* (1999)

split. This is potentially very important for the treatment of inheritance tax with large prizes. The situation is modeled in Figure 3. Our aspiring millionaire puts up $100,000 along with nine other friends for the $1 million bankroll and when they reach $10 million each share is worth $1 million. The syndicate must play full Kelly and has a chance of success of nearly 50 per cent assuming that the members agree to disband if they lose half their stake. Participants do not need to put up the whole $100,000 at the start. The cash outflow is easy to fund, namely 10 cents per draw per participant. To have a 50 per cent chance of reaching the $1 million goal, each participant (and their heirs) must have $50,000 at risk. It will take 22 million years on average to achieve the goal.

The situation is improved for case B players. First, the bankroll needed is about $154,000 since 65 tickets are purchased per draw for a $10 million wealth level. Suppose our aspiring nouveau riche is satisfied with $500,000 and is willing to put all but $25,000/2 or $12,500 of the $154,000 at risk. With one partner he can play half Kelly strategy and buy one ticket per case B type draw. Figure 4 indicates that the probability of success is about 0.95. With initial wealth of $308,000 and full Kelly it would take a million years on average to achieve this goal. With half Kelly it would take 2.7 million years and with quarter Kelly it would take 300 million years.

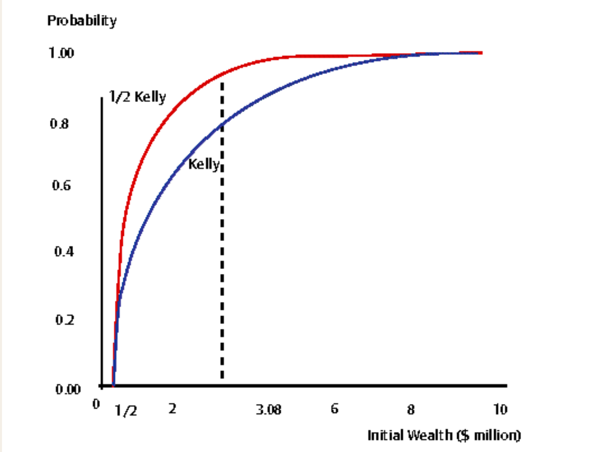
The conclusion is that except for millionaires and pooled syndicates, it is not possible to use the unpopular numbers in a scientific way to beat the lotto and have high confidence of becoming rich; these aspiring millionaires will also most likely be residing in a cemetery when their distant heirs finally reach the goal.

FIGURE 3: PROBABILITY OF REACHING THE GOAL OF $10 MILLION BEFORE FALLING TO $1 MILLION WITH VARIOUS INITIAL WEALTH LEVELS FOR KELLY, 1/4 KELLY AND 1/4 KELLY WAGERING STRATEGIES FOR CASE A



Source: *MacLean and Ziemba* (1999)

FIGURE 4: PROBABILITY OF REACHING THE GOAL OF $10 MILLION BEFORE FALLING TO $25,000 WITH VARIOUS INITIAL WEALTH LEVELS FOR KELLY AND 1//2 KELLY WAGERING STRATEGIES FOR CASE B



Source: *MacLean and Ziemba* (1999)

What did we learn from this exercise?

1. Lotto games are in principle beatable but the Kelly and fractional Kelly wagers are so small that it takes virtually forever to have high confidence of winning. Of course, you could win earlier and you have a positive mean on all bets. My studies have shown that the largest jackpots contain about 47 per cent of the nineteen most unpopular numbers in 1986 shown in Table 2 versus 17 per cent unpopular numbers in the smallest jackpots. Hence, if you play, emphasizing unpopular numbers is a valuable strategy to employ. Could you bet more? Sorry: log is the most one should ever bet as argued in my last column.

2. The Kelly and fractional Kelly wagering schemes are very useful in practice but the size of the wagers will vary from very tiny to enormous bets. My best advice: never over bet; it will eventually lead to trouble unless it is controlled somehow and that is hard to do!

FOOTNOTES & REFERENCES

1. Another is to look for lottery design errors. As a consultant on lottery design for the past twenty years, I have seen plenty of these. My work has been largely to get these bugs out before the games go to market and to minimize the damage when one escapes the lottery commissions' analysis. Design errors are often associated with departures from the pure pari-mutuel method, for example guaranteeing the value of smaller prizes at too high a level and not having the games checked by an expert.

2. See Ziemba et al (1986), *Dr Z´s Lotto 6/49 Guidebook*. While parts of the guidebook are dated, the concepts, conclusions, and most of the text provide a good treatment of such games. For those who want more theory, see MacLean and Ziemba, Growth versus security tradeoffs in dynamic investment analysis, *Annals of Operations Research* 85 (1999): 193-225 available from Kluwer or the author.

3. An estimate of the number of tickets sold versus the carryover in millions is proportional to the carryover to the power 0.811. Hence, the growth is close to 1:1 linear.