Bill Ziemba

The Capital Growth Theory of Investment: Part I

The use of log utility dates at least to the letters of Daniel Bernoulli more than two hundred years ago. The idea that additional wealth is worth less and less as it increases and this utility tails off proportional to the level of wealth is very reasonable to many students of investment. On the surface, this utility function seems safe for investing. However, I shall argue that log is the most risky utility function one should ever consider using and it is most dangerous. However, if used properly in situations where it is appropriate, it has wonderful properties. For long term investors who make many short term decisions, it yields the highest long run levels of wealth. This is called Kelly betting in honor of Kelly's 1956 paper that introduced this type of betting. In finance, it is often called the Capital Growth Theory.[1]

Consider the example described in Table 1. There are five possible investments and if we bet on any of them, we always have a 14% advantage. The difference between them is that some have a higher chance of winning and, for some, this chance is smaller. For the latter, we receive higher odds if we win than for the former. But we always receive \$1.14 for each \$1 bet on average. Hence we have a favorable game. The optimal expected log utility bet with one asset (here we either win or lose the bet) equals the edge divided by the odds.2 So for the 1-1 odds bet, the wager is 14% of one's fortune and at 5-1 it's only 2.8%. We bet more when the chance that we will lose our bet is smaller. Also we bet more when the edge is higher. The bet is linear in the edge so doubling the edge doubles the optimal bet.

However, the bet is non-linear in the chance of losing our money, which is reinvested so the size of the wager depends more on the chance of losing and less on the edge.

The simulation results shown in Table 2 assume that the investor's initial wealth is \$1000 and that there are 700 investment decision points. The simulation was repeated 1000 times. The numbers in Table 2 are the number of times out of the possible 1000 that each particular goal was reached. The first line is with log or Kelly betting, The second line is half Kelly betting. That is you compute the optimal Kelly wager but then blend it with cash. We will discuss later various Kelly fractions and how to utilize them wisely but for now, we will just focus on half Kelly. With lognormal (in continuous time) or normally distributed assets (in discrete time), the α-fractional Kelly wager is equivalent to the optimal bet obtained from using the concave risk averse, negative power utility function

$$\beta w^{\beta}, \ \beta < 0$$

where

$$\alpha = \frac{1}{1-\beta}.$$

For half Kelly (α=1/2), β = -1 and the utility function is w^{-1}. Here the marginal increase in wealth drops off as w^2, which is more conservative than log's w. Log utility is the case $\beta \to 0$ and cash is $\beta \to \infty$.

A major advantage of log utility betting is the 166 in the last column. In fully 16.6% of the 1000 cases in the simulation, the final wealth is more than 100 times as much as the initial wealth. Also in 302 cases, the final wealth is more than 50 times the initial wealth. This huge growth in final wealth for log is not shared by the half Kelly strategies, which have only 1 and 30, respectively, for their 50 and 100 time growth levels. Indeed, log provides an enormous growth rate but at a price, namely a very high volatility of wealth levels. That is, the final wealth is very likely to be higher than with other strategies, but the ride will be very very bumpy. The maximum, mean, and median statistics in Table 2 illustrate the enormous gains that log utility strategies can provide.

Let's now focus on bad outcomes. The first column provides the following remarkable fact: one can make 700 independent bets of which the chance of winning is at least 19% and usually is much more, having a 14% advantage on each bet and still turn £1000 into £18, a loss of more than 98%. Half Kelly has a 99% chance of not losing more than half the wealth versus only 91.6% for Kelly. The chance of not being ahead is almost three times as large for full versus half Kelly. Hence to protect ourselves from bad scenario outcomes, we need to lower our bets and diversify across many independent investments. This I will explore more fully in the context of hedge funds in the fifth and sixth columns of this series on the capital growth theory of investment.

Figure 1 provides a visual representation of the type of information in Table 2 displaying typical behavior of full Kelly versus half Kelly wagering in a real situation. These are bets on the Kentucky Derby from 1934 to 1998 using an inefficient market system where probabilities from a simple market (win) are used in a more complex market (place and show) coupled with a breeding filter rule [dosage filter 4.00] to eliminate horses who do not have enough stamina. Basically, you bet on horses that have the stamina to finish first, second or third who are underbet to come in second or better or third or better relative to their true chances estimated from their odds to win.

The full Kelly log bettor has the most total wealth at the horizon but has the most bumpy ride: $2500 becomes $16,861. The half Kelly bettor ends up with much less, $6945 but has a much smoother ride. The system did provide out of sample profits. A comparison with random betting proxied by betting on the favorite in the race, shows how tough it is to win at horseracing with the 16% track take plus breakage (rounding payoffs down) at Churchill Downs. Betting on the favorite turns $2500 into $480. Actual random betting has even lower final wealth at the horizon since favorites are underbet.

The difference between full and fractional Kelly investing and the resulting size of the optimal investment bets is illustrated via a tradeoff of growth versus security. This is akin to the static mean versus variance so often used in portfolio management and yields two dimensional graphs that aid in the investment decision making process. This can be illustrated by the game of blackjack where fractional Kelly strategies have been used by professional players.

The game of blackjack or 21 evolved from several related card games in the 19th century. It became fashionable during World War I and now has enormous popularity, and is played by millions of people in casinos around the world. Billions of dollars are lost each year by people playing the game in Las Vegas alone. A small number of professionals and advanced amateurs, using various methods such as card counting, are able to beat the game. The object is to reach, or be close to, twenty-one with two or more cards. Scores above twenty-one are said to bust or lose. Cards two to ten are worth their face value: Jacks, Queens and Kings are worth ten points and Aces are worth one or eleven at the player's choice. The game is called blackjack because an ace and a ten-valued card was paid three for two and an additional bonus accrued if the two cards were the Ace of Spades and the Jack of Spades or Clubs. While this extra bonus has been dropped by current casinos, the name has stuck. Dealers normally

TABLE 1: THE INVESTMENTS

Probability of Winning	Odds	Probability of Being Chosen in the Simulation at Each Decision Point	Optimal Kelly Bets Fraction of Current Wealth
0.57	1-1	0.1	0.14
0.38	2-1	0.3	0.07
0.285	3-1	0.3	0.047
0.228	4-1	0.2	0.035
0.19	5-1	0.1	0.028

TABLE 2: STATISTICS OF THE SIMULATION

Final Wealth					Number of times the final wealth out of 1000 trials was				
Strategy	Minimum	Maximum	Mean	Median	>500	>1000	>10,000	>50,000	>100,000
Kelly	18	483,883	48,135	17,269	916	870	598	302	166
Half Kelly	145	111,770	13,069	8,043	990	954	480	30	1

Source: *Ziemba and Hausch, Betting at the Racetrack* (1986)

FIGURE 1:

Wealth level histories from place and show betting on the Kentucky Derby, 1934-1998 with the Dr Z system utilizing a 4.00 dosage index filter rule with full and half Kelly wagering from $200 flat bets on the favorite using an initial wealth of $2500. *Source: Bain, Hausch and Ziemba (1998)*

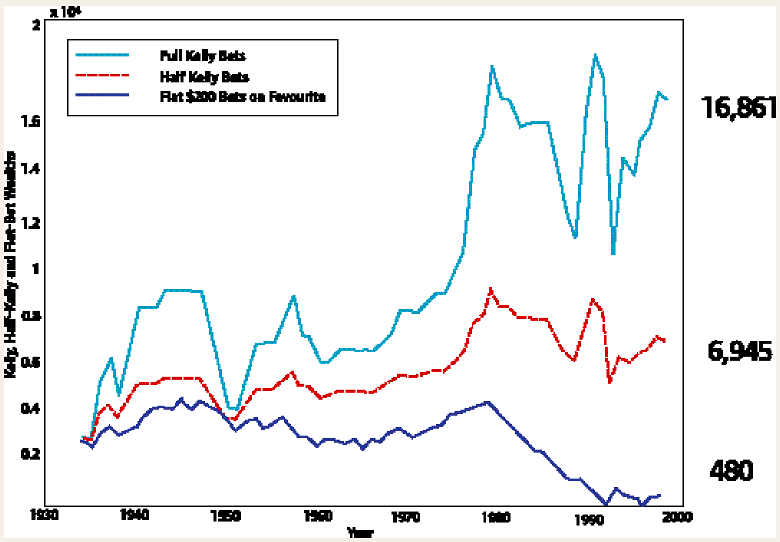

FIGURE 2:

Probability of doubling and quadrupling before halving and relative growth rates versus fraction of wealth wagered for Blackjack (2% advantage, p=0.51 and q=0.49). Source: *McLean and Ziemba* (1999)

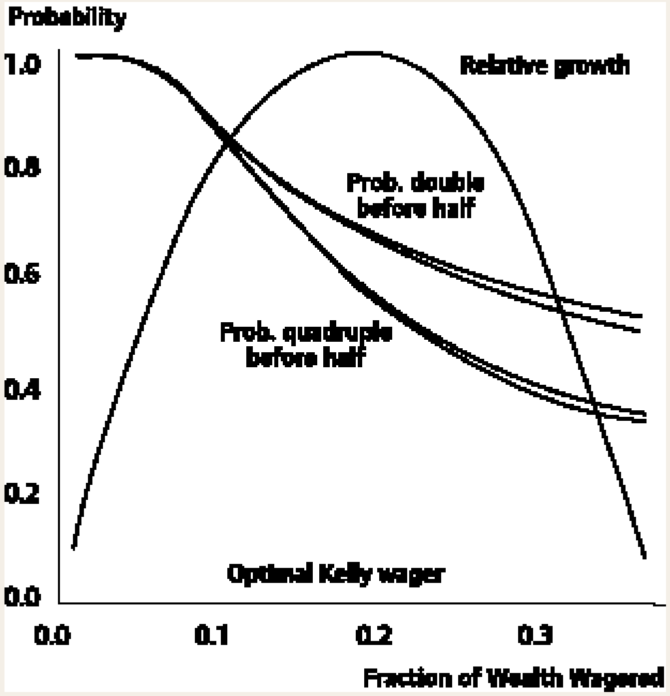

TABLE 3: GROWTH RATES VERSUS PROBABILITY OF DOUBLING BEFORE HALVING FOR BLACKJACK

	Kelly Fraction of Current Wealth		P[Doubling before Halving]		Relative Growth Rate
	0.1		0.999		0.19
	0.2		0.998		0.36
Range	0.3		0.98		0.51
for	0.4	Safer	0.94	More Growth	0.64
Blackjack	0.5	Half Kelly	0.89		0.75
Teams	0.6	Riskier	0.83	Less Growth	0.84
	0.7		0.78		0.91
	0.8		0.74		0.96
	0.9		0.70		0.99
Overkill ǂ Too Risky	1.0	Full Kelly	0.67		1.00
	1.5		0.56		0.75
	2.0	Zero Growth	0.50		0.00

Source: *MacLean and Ziemba* (1999)

play a fixed strategy of drawing cards until the total reaches seventeen or more at which point they stop. A variation is when a soft seventeen (an ace with cards totaling six) is hit. It is better for the player if the dealer stands on soft seventeen. The house has an edge of 1-10% against typical players. The strategy of mimicking the dealer loses about 8% because the player must hit first and busts about 28% of the time (0.282 ≈ 0.08). However, in Las Vegas the average player loses only about 1.5% per play.

The edge for a successful card counter varies from about -5% to +10% depending upon the favorability of the deck. By wagering more in favorable situations and less or nothing when the deck is unfavorable, an average weighted edge is about 2%. An approximation to provide insight into the long-run behavior of a player's fortune is to assume that the game is a Bernoulli trial with a probability of success $p = 0.51$ and probability of loss $q = 1 - p = 0.49$.

Figure 2 shows the relative growth rate $\pi \ln(1+p) + (1-\pi)\ln(1-\pi)$ versus the fraction of the investor's wealth wagered, π. This is maximized by the Kelly log bet $\pi^* = p - q = 0.02$. The growth rate is lower for smaller and for larger bets than the Kelly bet. Superimposed on this graph is also the probability that the investor doubles or quadruples the initial wealth before losing half of this initial wealth. Since the growth rate and the security are both decreasing for $\pi > \pi^*$, it follows that it is never advisable to wager more than π^*. Also it can be shown that the growth rate of a bet that is exactly twice the Kelly bet, namely $2\pi^* = 0.04$, is zero. Figure 2 illustrates this. Hence log betting is the most aggressive investing that one should ever consider. The root of hedge fund disasters is frequently caused by bets above π^* when they should have bets that are usually below π^*, especially when parameter uncertainty is considered. However, one may wish to trade off lower growth for more security using a fractional Kelly strategy. This growth tradeoff is further illustrated in Table 3. For example, a drop from _* = 0.02 to 0.01 for a 0.5 fractional Kelly strategy, decreases the growth rate by 25%, but increases the chance of doubling before halving from 67% to 89%.

In the next four columns, I will discuss three topics: investing using unpopular numbers in lotto games with low probabilities of success where the returns are very large (this illustrates how bets can be very tiny) next issue; good and bad properties of the Kelly log strategy and why this led me to work with Len MacLean on a thorough study of fractional Kelly strategies and futures and commodity trading, and how large undiversified positions can lead to disasters as it has for numerous hedge funds and bank trading departments in the fifth and sixth columns of this series.

FOOTNOTES & REFERENCES

1 For readers who would like a techinical survey of capital growth theory, see Hakansson and Ziemba's article in *Finance*, eds, Jarrow, Maksimovic and Ziemba eds, North Holland, 1995.

2 For one or two assets with fixed odds, take derivatives and solve for the optimal wagers; for multi-asset bets under constraints; and when portfolio choices affect returns (odds), one must solve a nonlinear program which, possibly, is non-convex.