Poker as a Lottery

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Introduction

Doyle Brunson[1], two-time winner of the World Series of Poker main event, has likened a poker tournament to a lottery in which more skilled players (like himself) hold more tickets than less skilled players (like myself). In this article, I work out the details of this analogy and provide some very general and, I think, very important results for anyone hoping to be a winning poker player.

Lottery Analytics

Consider a lottery in which each player is allowed to write his name on any number of tickets and drop them into a hat. After each player has deposited his tickets, the hat is shaken. The winner's name is drawn first. The drawing then continues until a distinct name (not the winner's) is drawn; that player is the second place finisher. The drawing then continues again until a third distinct name (not the winner's and not the second place finisher's) is drawn; that player is the third place finisher. This procedure is repeated until each player's name is drawn, the random order of names determining the placement of each player.

Let's say the lottery has N players. One of the players, named Doyle coincidently, writes his name on α tickets; all the other players write their names on β tickets each. The probability that Doyle finishes in n^{th} place is equal to the conditional probability that he places n^{th}, given that he hasn't placed, times the probability that he hasn't placed:

$$p_n = \frac{\alpha}{(N-n)\beta + \alpha}\left[1 - \sum_{m=1}^{n-1} p_m\right]$$

The conditional probability's numerator is Doyle's number of tickets; its denominator is the total number of tickets remaining, given that Doyle hasn't placed. p_n is really just a function of the ratio α/β, or, better still, of $s \equiv \alpha/\beta - 1$, which measures Doyle's advantage. Table 1 shows p_n for $N = 10$ and various s.

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Poker Analytics

Consider now a poker tournament into which N players pay x_0 and from which the nth place finisher is paid x_n. The buy-in x_0 is composed of two pieces: the notional q, which goes into the prize pool and is later distributed to the tournament winners, and the fee (or "vig") q', which goes to the tournament organizer; $x_0 = q + q'$. Such tournaments are available on the internet starting every few minutes. Table 2 shows typical values for q and q'. Table 3 shows typical values for N and $x_n/(Nq)$.

A priori, poker skill is a complex function of a player's strategy. Acquiring skill is the subject of most articles and books on poker. In this article, I take a different approach. Guided by Brunson's analogy that a poker tournament is like a lottery, I equate poker skill to the number of tickets held in a lottery and define an *a posteriori* method for measuring poker skill.

In an N player poker tournament, one player, again named Doyle as homage, has skill α; all other players have skill β. As in the lottery, Doyle's advantage is given by his *relative* skill $s \equiv \alpha/\beta - 1$. This is an important point: even an excellent poker player (in an absolute sense) is at a disadvantage when playing against players who are better than he is. Further, if all players use the same strategy, even the game theoretically optimal (Nash equilibrium) strategy, no one has an advantage.

Doyle's payout from one poker tournament is a random variable, X let's say, and his profit is another random variable, $Y \equiv X - x_0$. Both X and Y, through p_n, are functions of s. The mean and variance of Y, also functions of s, are

$$\mu = \langle Y \rangle$$
$$\sigma^2 = \langle Y^2 \rangle - \langle Y \rangle^2$$

where

$$\langle f(Y) \rangle \equiv \sum_{n=1}^{N} p_n f(y_n)$$

TABLE 1: PROBABILITY OF PLACING

P_n			s	
		−50%	**0%**	**50%**
	0			
n	1	5%	10%	11%
	2	6%	10%	11%
	3	6%	10%	11%
	4	6%	10%	11%
	5	7%	10%	10%
	6	8%	10%	10%
	7	9%	10%	10%
	8	11%	10%	9%
	9	14%	10%	9%
	10	28%	10%	8%

TABLE 2: TYPICAL BUY-INS

x_0 [$]	q [$]	q' [$]
6	5	1
11	10	1
22	20	2
33	30	3
55	50	5
77	70	7
109	100	9
215	200	15
530	500	30

TABLE 3: TYPICAL PAYOUTS

$x_n/(Nq)$				N			
		2	**6**	**10**	**20**	**30**	**50**
n	1	100%	65%	50%	40%	30%	30%
	2		35%	30%	30%	25%	22%
	3			20%	20%	20%	17%
	4				10%	15%	12%
	5					10%	9%
	6						6%
	7						4%

Figure 1 shows the distribution of Y for a ten player, $10 + $1 tournament for $s = 0$ and $s = 50\%$.

If Doyle plays in a series of I, N player, x_0 buy-in tournaments against opponents who all have the same skill level, his profit per tournament is the random variable $Y_I \equiv Y/I$. The mean and standard deviation of Y_I are

$$\mu_I = \mu$$
$$\sigma_I = \frac{\sigma}{\sqrt{I}}$$

Figures 2 and 3 show for $I = 10$ and $I = 20$, the evolution of Y_I for a series of ten player, $10 + $1 tournaments for $s = 0$ and $s = 50\%$. As you can see, the more tournaments Doyle plays, the closer the distribution gets to the normal distribution (which is the Central Limit Theorem).

Now let's say Doyle, starting with an initial bankroll b_0, actually plays in I, N player, x_0 buy-in tournaments. He wins some and he loses some and in the end he's grown (or pruned!) his bankroll to b_I, for an average profit per tournament of $y_I = (b_I - b_0)/I$. The maximum likelihood estimate of Doyle's skill s is given implicitly by $\mu(s) = y_I$. Further, in order to reject, with γ confidence, the null hypothesis, $H_0: s = 0$, that Doyle is an unskilled player and accept the alternative hypothesis, $H_1: s > 0$, that he
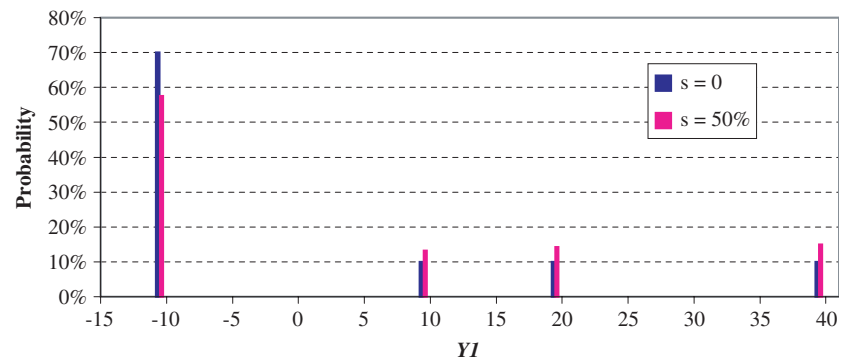

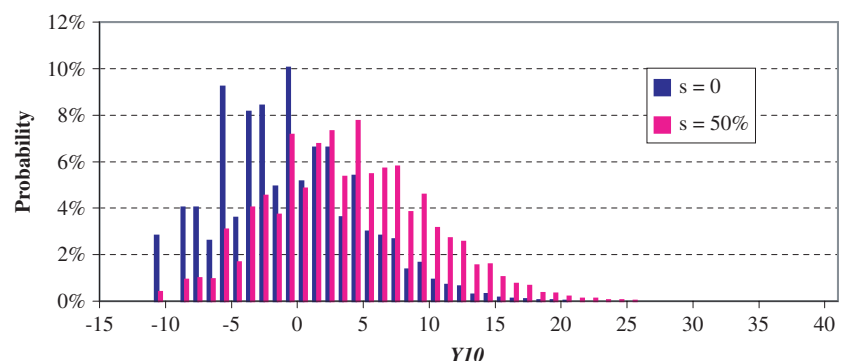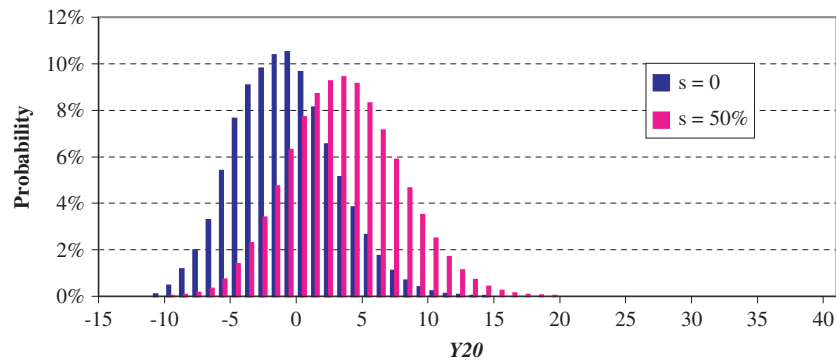**Figure 1: Profit Distributions after 1 Tournament.**

Figure 2: Profit Distributions after 10 Tournaments.

Figure 3: Profit Distributions after 20 Tournaments.

is a skilled player, Doyle must earn an average profit per tournament of at least y^*:

$$y^* = \frac{\sigma}{\sqrt{I}} \Phi^{-1}(\gamma) + \mu$$

where μ and σ are calculated here with $s = 0$ and Φ^{-1} is the inverse standard normal cumulative distribution function. Here, we assume Doyle has played enough tournaments (in practice 10 to 20) that the distribution of Y_I is very closely normal. For example, for a series of 20, ten player, $10 + $1 tournaments, $y^* = 5.15 at the 95% confidence level.

Lastly, let's say, with γ confidence, that Doyle has statistically significant skill s. If his bankroll is b_I and he plays in an x_0 buy-in tournament, his bankroll growth rate, a random variable, is

$$R = \ln\left(1 + \frac{Y}{b_I}\right)$$

and his expected bankroll growth rate, his expected return, is

$$r \equiv \langle R \rangle = \sum_{n=1}^{N} p_n \ln\left(1 + \frac{y_n}{b_I}\right)$$

TABLE 4: OPTIMAL BUY-IN FRACTIONS

f^*		N					
		2	**6**	**10**	**20**	**30**	**50**
s	0%	−11%	−5%	−4%	−2%	−2%	−1%
	25%	1%	4%	4%	2%	2%	2%
	50%	11%	11%	10%	7%	6%	4%

The Kelly Criterion[2] says that Doyle should choose a buy-in x^* that maximizes his expected return, given his bankroll b_I. Table 4 shows the optimal buy-in fraction, $f^* \equiv x^*/b_I$, for players of various skill levels playing in tournaments with a 10% vig (q'/q) and the typical payout structures of Table 3. (A negative optimal buy-in fraction means the player does not have enough skill to overcome the vig.) A player, even one with statistically significant skill s, should not buy into a tournament too large for his bankroll and should, instead, choose the largest tournament buy-in x_0 available for which $x_0/b_I < f^*(s)$.

Conclusion

The analysis presented here views a poker tournament as a lottery in which poker skill is equated to the number of lottery tickets held. No mention is made of the details of the player's poker strategy. One result of this analysis is that relative skill determines poker success. The second result is a statistical methodology for measuring poker skill. The third result is a recipe for maximizing bankroll growth, given statistically significant skill.

FOOTNOTES

1. Brunson, D. (2005). Televised Interview, "The 2005 World Series of Poker", *ESPN*.
2. Kelly, J. L., Jr. (1956). "A New Interpretation of the Information Rate", *Bell System Technical Journal*, 917–926.