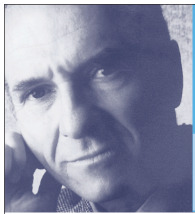
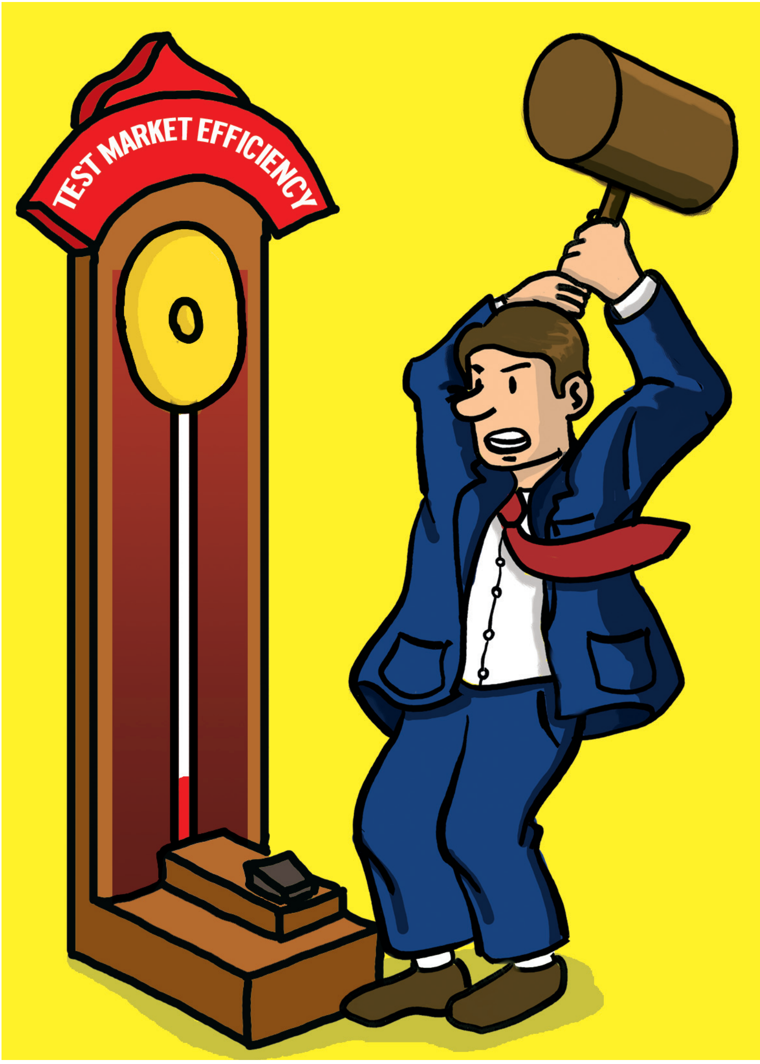


Ed Thorp

A MATHEMATICIAN ON WALL STREET

Inefficient Markets

The markets are far more efficient when viewed from the banks of the Charles than from the banks of the Hudson.
Fischer Black



The "crash of '87" was the most extreme stock market price jump of the twentieth century. The S&P 500 Index fell over 20 per cent in one trading day, measured by the change in closing prices from Friday, October 16, 1987 to the close the following Monday, October 19. But if the market were close to efficient then both these closing prices must be, to good approximation, "correct." Let's see what this implies.

Suppose that every security, including not only individual issues but "portfolios," has at any time t a "correct" or "true" price $Q(t)$, a current market price $P(t)$, and a deviation $D(t)$ of the market price from the correct price, satisfying the equation $P(t) = Q(t) + D(t)$. If the bid-asked spread and transactions costs are "small," then to a good approximation $P(t)$ is an observable number. Since we'll be concerned with the relative sizes of $Q(t)$ and $D(t)$, it will be useful to consider the true price and deviation as a proportion of $P(t)$. Dividing through by $P(t)$ gives $I(t) = q(t) + d(t)$, where $I(t)$ is one unit of the security and $q(t) = Q(t)/P(t)$ is the portion corresponding to the true price and $d(t)$ is the difference between one unit and the true price portion. If a security is efficiently priced at time t then $d(t)$ is very small compared to $I(t)$ and $q(t)$. So, assuming market efficiency, the correct price of the S&P 500 Index changed by approximately 20 per cent in one day. Yet an analysis by Shiller (1987) finds no news or information based explanation either ex ante or ex post. This implies that the one-day change in the correct price must have been much less than 20 per cent. For discussion purposes, suppose that it were "just" a "two sigma event." This typically is a change in the index of about 2 per cent. Then we have about 18 per cent left over as the sum of the two deviations d(Friday close) and d(Monday close). The best we can do from a min max point of view is to put this 2 per cent band of correct pricing midway between the two closing prices, which allocates a 9 per cent error to each closing price. We conclude that the S&P 500 was mispriced by at least 9 per cent at one or both of the two closes. Since the whole US market and markets worldwide behaved similarly, we're looking at a minimum aggregate mispricing of $200 bn or so in the US and a comparable additional amount worldwide.

Absolute and relative mispricing

A security is *absolutely mispriced* at time t if $I(t) \neq q(t)$, i.e. $d(t) \neq 0$. We're of course only concerned throughout this article with "significant" deviations from zero, as everyone accepts the fact that there is a small irreducible random "chatter" of $d(t)$ around zero and that this doesn't violate the spirit of market efficiency. For an illustration of how this chatter can violate market efficiency and produce excess returns, see my "Statistical Arbitrage," Parts I-VI, in *Wilmott* Sept. '04–July '05. We argued that the market as a whole was absolutely mispriced by at least 9 per cent at the close on at least one of the two days Friday, October 16, 1987 or Monday 19. However we can't tell from our reasoning on which of the two days this occurred and how much greater the mispricing might have been.

We used a relative mispricing argument for our deduction. A pair of securities is relatively mispriced if, given $I_i(t) = q_i(t) + d_i(t), \quad i = 1, 2$, we have

$$I_1(t) - I_2(t) \neq q_1(t) - q_2(t)$$

or, equivalently, $d_1 - d_2 \neq 0$. It follows that if two securities are relatively mispriced, either $d_1 \neq 0$, $d_2 \neq 0$, or both, so at least one of the securities must be absolutely mispriced. The relative mispricing is $d = d_1 - d_2$ hence at least one of d_1 or d_2 must satisfy $|d_i| \geq |d|/2$ so the magnitude of the absolute mispricing for one or both of the pair must be at least $|d|/2$. We applied this general argument to the crash of '87 with one additional assumption: there we compared securities at two different times and had to use informational arguments to tie them together. At the end of this article we'll give an example without this additional step. It is similarly extreme but uses simultaneously priced securities.

Micro versus macro efficiency

If we form a weighted average, e.g. an index, of individual securities, we intuitively would expect some "cancellation" of their absolute mispricing with the consequence that the absolute mispricing of the index tends to be less than the absolute mispricing of the components. To see this mathematically, suppose

$$I_i(t) = q_i(t) + d_i(t), i = 1, \ldots, n$$

and that we form the index $I_M(t) = \sum_{i=1}^n a_i I_i(t)$ where the a_i are non-negative weights with $\sum_{i=1}^n a_i = 1$. Then $I_M(t) = \sum_{i=1}^n a_i q_i(t) + \sum_{i=1}^n a_i d_i(t)$ and if we assume that $\sum_{i=1}^n a_i q_i(t) = q_M(t)$, we have $d_M(t) = \sum_{i=1}^n a_i d_i(t)$, from which the triangle inequality gives $|d_M(t)| \leq \sum_{i=1}^n a_i |d_i(t)|$, i.e. the absolute mispricing of the index is less than or equal to the weighted average of the absolute values of the absolute mispricing of the individual securities.

This suggests that inefficiencies are greater among individual securities than with the market as a whole. On the other hand we have what Jung and Shiller (2002), quoting from a private letter from Samuelson, call Samuelson's Dictum:

"Modern markets show considerable *micro* efficiency (for the reason that the minority who spot aberrations from micro efficiency can make money from those occurrences and, in doing so, they tend to wipe out any persistent inefficiencies). In no contradiction to the previous sentence, I had hypothesized considerable *macro* inefficiency, the sense of long waves in the time series of aggregate indexes of security prices below and above various definitions of fundamental values."

They add (2005) that this means "the efficient markets hypothesis works much better for individual stocks than it does for the aggregate stock market." They then go on to review evidence in recent literature and also test stock market data, both supporting the Dictum and seeming to contradict the triangle inequality!

I believe this is resolved using the distinction between absolute and relative mispricing. To illustrate with an extreme example, if the market were absolutely mispriced (absolutely macro inefficient) but pairs of individual securities were not relatively mispriced (relatively micro efficient) then each pair of securities would satisfy $d_i(t) = d_j(t)$ hence $d_i(t) = c$, a constant, for $i = 1, \ldots, n$. Then $d_M(t) = \sum_{i=1}^n a_i d_i(t) = c$ as well and macro inefficiency holds if $c \neq 0$. In other words, all securities are mispriced by the same percentage so the market is mispriced but there is no relative mispricing between securities. *Absolute* macro mispricing (macro inefficiency) of markets seems evident to the casual observer, such as the 1979-81 interest rate and precious metals price spikes, the crash of '87, the dot com "bubble," and current housing prices in large parts of both the US and the rest of the world. Exploitation? Perhaps by asset reallocation. Note that asset reallocation exploits the relative mispricing between asset classes but not their absolute mispricing.

That covers the first part of the dictum. What about micro efficiency? If Samuelson and Shiller and Jung mean *relative* mispricing, as I believe they do, then there is no contradiction and what the dictum is telling us is that nearly all the absolute mispricing of individual stocks is due to the absolute mispricing of the overall market, with just a minor amount due to relative mispricing of individual securities.

Just how good is this claimed micro efficiency? We've given general examples in this column, such as the stories of statistical arbitrage and convertible hedging, and the specific example of the COMS/PALM spinoff. For more on this and other mispriced spinoffs, see Lamont and Thaler (2003).

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Derivatives theorists will be amused (if they don't believe the EMH) by another example, the price of Redback Networks (RBAK) compared to two of its warrants. In an extreme contradiction to rational warrant pricing, both warrants traded at prices substantially above the price of the stock. Details: The terms for RBAKZ were one warrant + \$9.50 can buy one share of RBAK until Jan. 2, 2011. Similarly one RBAKW warrant + \$5.00 can buy one share of RBAK until the same date. For almost all the first four months of 2004, the price of RBAKW exceeded that of RBAK. The same was generally true for RBAKZ as well. On Feb. 5, 2004, for example, the prices were RBAK \$8.30, RBAKW \$12.50 and RBAKZ \$15.15! As I've made a living for 38 years by exploiting relative micro mispricing, its magnitude and extent are of great interest to me.

Arnott's argument

Arnott, Hsu and Moore (2004) and Arnott (2005) develop an idea for estimating the amount of *relative* micro inefficiency in the market. Here's the root idea. Suppose that at the start of each period each security has probability 1/3 of being in each of three states: (1) $P_i = (1+a)Q_i$, (2) $P_i = Q_i$, or (3) $P_i = (1-a)Q_i$. At the beginning of the next period the state is independent of the state in the prior period. Securities are on average fairly priced (absolute macro efficiency) but individually fluctuate randomly around their unknown true price. (The idea works just as well with absolute macro inefficiency but the details are more complicated.) Each state transition has probability 1/9. If we form an equally weighted portfolio and rebalance to equal weights at the start of each new period, a calculation shows that we gain $G = 2a^2/(3(1-a^2))$ per period. For instance, the transition from state (1) to state (3) changes an amount $1+a$ to $1-a$ for a loss per unit of $-2a/(1+a)$. The variance σ^2 per period of $d(t)$ is $2a^2/3$ so the gain per period can be written as $\sigma^2/(1-a^2)$. If $a = 1/6$, for instance, we have $G = 2/105$ or about 2%.

Arnott finds, under a number of scenarios that may exploit this effect – one of which is equal weighting, that he can get historical returns of about 2 per cent more than the market index without offsetting increases in risk. The results if due to this cause, suggest that the relative micro inefficiency may have order of magnitude of fourteen percent or more! Clearly the root idea here can be extended to more realistic probability distributions and transition probabilities, with essentially the same type of result. Hsu (2004) gives a general derivation where $d_i(t)$ is white noise with mean zero and variance per unit time σ^2. He finds in this case the expected value of the ratio I_A/I_M, where I_A is an equal weighted index satisfies

$$E(I_A/I_M) = \exp(\sigma^2(t_2 - t_1)),$$

i.e. the expected growth rate of I_A per unit time exceeds that of I_M by σ^2. Hsu's derivation assumes that $d_i(t), d_i(t+1), \ldots$ are independent, as does our example. Under these assumptions and the measured effect of about 2 per cent per year, $\sigma \doteq \sqrt{.02} = 14\%$.

This is likely to be quite an underestimate. Here's why. It's intuitive that regression of $d(t)$ towards the mean ought to have some associated characteristic time. Arnott et al. find that the mispricing effect doesn't vary much if rebalancing is done quarterly, semi-annually or annually. This suggests that for these time intervals there are substantial positive correlations between successive $d(t)$, $d(t+1)$, etc. But then it turns out that as ρ increases from zero, a larger σ^2 is required to produce a given effect, hence the implication that the average relative micro mispricing is likely to be considerably larger than 14%. I suspect that $E(I_A/I_M) = \exp((1-\rho)\sigma^2(t_2 - t_1))$ for $\rho \neq 0$, with ρ perhaps of the form $\rho = \exp(-k(t_2 - t_1))$. For $\rho = 1/2$ and $t_2 - t_1 = 1$ year, this gives $\sigma^2 = 0.04$ or $\sigma = 0.20$, up from $\sigma = 0.14$ when $\rho = 0$.

The Crash of '87, Day 2

The day after the 20 per cent drop in the S&P 500, I observed the S&P futures contract trading at about 190 and the S&P index trading at about 220, for a relative macro mispricing of more than 10per cent. As experienced index arbitrageurs, we at Princeton Newport Partners knew that, ordinarily, the two should and did satisfy "no arbitrage" conditions to within a fraction of a percent. We therefore shorted a little more than 10 million dollars worth of a diversified basket of stocks and bought 10 million dollars worth of index futures, realizing a gain of more than a million dollars when the relationship returned to nearly normal. One or both of these two securities, by our earlier argument, had to be absolutely (macro) mispriced by at least 5 per cent.

We have developed a framework for thinking about market inefficiencies. In addition to the distinctions between absolute and relative mispricing, and between macro and micro inefficiency, we see that the total market value and extent of these inefficiencies appears substantial. However much of this isn't, and perhaps may never be, linked to specific securities, i.e. it exists but is not "observable." Further, much of what is observable is not exploitable due to market defects, costs, and the tendency of the mispricing to diminish as a consequence of the trades that exploit it.

As Steve Ross observed, the total market value of the available alpha is generally far less than the total market value of the alpha that exists. Nevertheless, fortunes have been and will continue to be made by extracting the alpha that is available.

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