Naïve low volatility equity portfolios are risky

A practical study in successful implementation

July, 2011*

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**Submitted entry author date, this version subsequently revised to include background on model employed in research findings.*

Executive Summary

- Low volatility (LV) equity portfolios are long-only equity portfolios built to have as little volatility as possible.

- The essence of LV strategies is smaller but more consistent returns.

- Traditional equity investing wisdom is based on the Security Market Line, i.e. more risk will be correlated with more return. However it can be demonstrated that a low risk portfolio can beat the market index on both a risk (standard deviation) and return basis.

- Equity indices are simply constructed based on arbitrary membership criteria and market capitalization. Traditional indices are not mean-variance efficient, and do not lie on the efficient frontier.

- A simple "naïve" approach to constructing minimum variance strategies can be prohibitive for a number of reasons and sub-optimal at best. LV strategies can be enhanced and significantly improved by adding quality screens on the universe.

- A statistical risk model produces better risk-adjusted performance vs. traditional fundamental-based risk model.

- This paper examines some of the portfolio construction techniques used to build portfolios that can deliver excess returns, while minimizing total risk (standard deviation).

Introduction:

The subprime mortgage-induced financial crisis and the global economic recession have pushed the world into a new regime in which "volatility" has taken on new meaning for risk-averse investors. Although the day that the S&P 500 traded at 666 seems a distant memory, 2011's devastating natural disasters in Japan, coupled with sovereign debt crises in the Eurozone and the "huge China bubble" theory, serve as reminders that the traditional risk-return theories, equity investing and normal distribution assumptions have all changed. Or, at least, they are not as reliable as they once were. Instead, minimum variance portfolios have quickly attracted attention in the equity world. They provide a refreshing concept of investing with less focus on performance and more emphasis on risk management. And better yet, they actually provide excellent "alpha" by simply focusing on the less risky portion of the market.

The question is: Is the approach to creating a minimum variance strategy as naïve as simply minimizing the overall risk of a portfolio, or is there more to it? In contrast to many existing studies, this paper examines different ways to create minimum variance portfolios by escalating the level of sophistication from a portfolio engineering perspective to shed light on the "spectrum" of min-var strategies. Hopefully, we can provide some guidance to help potential investors understand this popular portfolio concept and assist them in selecting the right products to best meet their requirements.

This paper is organized as follows:

Section I: Background of minimum variance portfolio and a brief discussion on the return anomaly with a simple naïve back test.

Section II: An empirical study of different types of Canadian minimum variance portfolios to improve on the naïve method and we compare their behaviors. In-sample period of August 2000 to January 2011 and an out-of-sample period of Feb, 2011 to Feb, 2012 are studied with difference of mean and variance analysis to statistically determine the best strategy's "alternativeness" to the common S&P/TSX Composite Index.

Section III: Apply similar back test strategies to the global equity market using MSCI World Total Return Index as the reference benchmark and then conduct the same statistical analysis on the best one.

Section IV: Conclusion and discussion.

Appendix 1: Back test data and statistical table **Appendix 2:** Axioma risk models and optimization

Section I

A base case and the theories

The traditional portfolio optimization process seeks the optimal weighting vector **W** of asset holdings in a portfolio by solving the following equation:

Maximize $\sum_{i=1}^{n} \boldsymbol{Wi} * \boldsymbol{E}(\boldsymbol{Ri}) - \lambda * \boldsymbol{\delta p}$

Subject to: Constraints

Where,

$E(Ri)$ are the forecasted returns, i.e. Manager alpha for stocks in the investment universe.

λ is the risk tolerance multiplier, δp is the estimated total portfolio volatility.

A minimum variance portfolio creation process, on the other hand, solves a different equation:

Minimize $\boldsymbol{\delta p}$

Subject to: Constraints

The significant difference between these two portfolio strategies is immediately obvious: The "alpha" portion of the portfolio construction process is ignored in the min-var strategy. The only factor the investor focuses on in this case is the volatility of the portfolio.

Research shows that in different geographic markets, minimum variance portfolios have unique return anomalies and outperform the market indices from both a risk and a return perspective. This is very counter-intuitive because traditional wisdom informs us that the security market line is steep, i.e. more risk will be correlated with more return. Therefore, how can the returns of low risk portfolios beat the market index? In seeking an answer, we started by conducting a simple minimum variance portfolio back test to verify this "anomaly". This serves as a foundation for understanding and exploring this type of strategy.

Back test #1's environment and parameters are:

- **Period:** August 2000 to January 2011. (In-sample)
- **Market:** Canadian equity market
- **Reference benchmark:** S&P/TSX Composite Index

- **Risk model:** Axioma Canadian Fundamental Risk Model[1]
- **Investment Universe:** Top 200 Canadian stocks and top 80 income trusts based on market capitalization
- **Objective:** Minimize the overall risk of the designed portfolio, rebalance monthly.

  Subject to the following constraints:

  GICS sector weightings: Maximum 30% each

  Income trust holdings: Maximum 15%

  Individual income trust names: Maximum 2% each

  (These two constraints are to mitigate the historical "income trust" effect)

  Axioma industry group weighting: Maximum 25% each

  Range for number of holdings: 40 to 80

  Minimum weight per holdings: 0.5%

  Maximum individual stock weights: 4%

  No turnover constraints

  Transaction costs: None for the back test

The results of the back test are shown in Table 1 and Exhibit 1

Exhibit 1 Hypothetical Results: Back Test #1

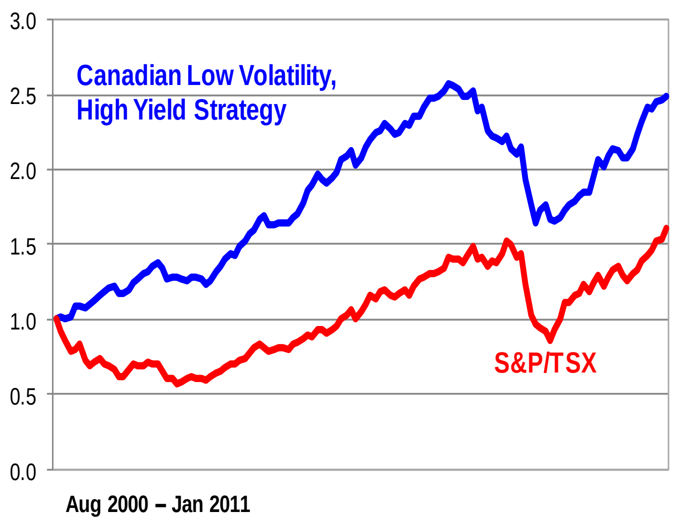


Table 1

Realized Results	Value
Annualized Return	9.09%
Average Annualized (Total) Risk	9.90%
Annualized Active Return	**4.49%**
Average Annualized Active Risk	12.10%
Average Period Turnover	**27.29%**
Average Number of Assets	62.6825
Sharpe Ratio	0.9184
Information Ratio	0.3712

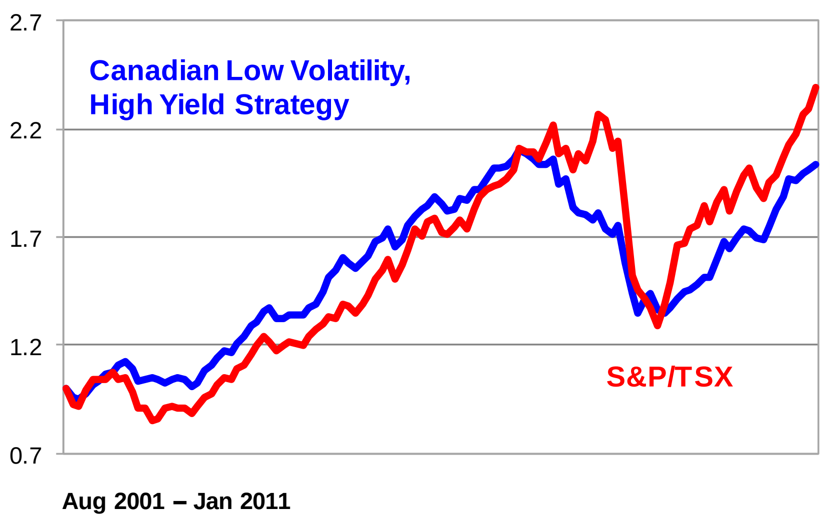
[1] See Appendix 2 for details on Axioma risk models and optimization process

At first glance:

- There is a sizable active return for this minimum variance portfolio compared to the S&P/TSX index in Canada (4.49% annualized).
- Given that turnover is not constrained, turnover averaged 27.29% per month, which is approximately 327% annualized.
- Transaction costs are not considered in this back test. However, if we assume a linear approximation on transaction costs, i.e. 1 basis point total cost per 100% turnover, we are looking at 3.27% being deducted from the excess performance. This leaves the strategy with an annualized excess return of only 1.22%. This isn't much after considering management fees and other expenses.
- During the early period of this back test, the strategy appears to have benefitted significantly from the technology bubble implosion. This is reflected in the cumulative returns.
- During the 2008 financial crisis, this minimum variance portfolio underperformed the index and led the index on the downside.

Given that the early period of this back test contributed so much to excess return, we would like to start our back test in Aug, 2001. Now back test #1 actually underperformed both during the financial crisis and the market recovery in 2009!

Exhibit 2 Hypothetical Results: Back Test #1



At this point, we would like to introduce an important statistical test that we will use throughout this research: A "difference of mean and variance" test to demonstrate whether these lower volatility strategies are truly different from the related equity indices from a mean and variance perspective. This is how the test proceeds:

1. Determine whether the two samples (A particular strategy monthly return series and the index return series) have equal variance or not, using F distribution statistics. (Note: one-tailed test, degree of freedom = N-1 for both samples and the bigger sample variance is on the numerator)

$$F_{n_1;n_2} = \frac{s_1^2}{s_2^2} \qquad \begin{aligned} H_0 &: \sigma_1^2 = \sigma_2^2 \\ H_1 &: \sigma_1^2 > \sigma_2^2 \end{aligned}$$

2. Once 1 is concluded and the two samples have same variance, the difference of mean test can then be performed using the following t statistics. (n1=n2)

$$t = \frac{\overline{X}_1 - \overline{X}_2}{\sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}}\sqrt{\frac{n_1 + n_2}{n_1 n_2}}}$$

Degree of freedom is:

$$n_1 + n_2 - 2$$

Of course, in our study, we would hope that this is not the case given the strategy is designed to come up with a significantly “lower volatility” equity solution.

For the case of variance inequality, the denominator of the t stats becomes:

$$\sqrt{\frac{s_1^2}{n_1 - 1} + \frac{s_2^2}{n_2 - 1}}$$

Degree of freedom is:

$$df = \frac{\left(\frac{s_1^2}{N_1 - 1} + \frac{s_2^2}{N_2 - 1}\right)^2}{\left(\frac{s_1^2}{N_1 - 1}\right)^2\left(\frac{1}{N_1 + 1}\right) + \left(\frac{s_2^2}{N_2 - 1}\right)^2\left(\frac{1}{N_2 + 1}\right)} - 2$$

Let us try this process with monthly return data in back test #1 for Aug 2001 to Jan 2011 (Appendix), where S1 is the variance of the monthly TSX return series and S2 is the variance of back test #1 monthly return. So the F stat = S1/S2 = 0.18%/0.08%=2.25. Degree of freedom is calculated to be: 126-1=125. Based on the F distribution table (In the Appendix) using 120 and 120 as the closest degree of freedom value for the two samples, we obtained the threshold value of 1.533 with 99% confidence level for the right tail test. Since 2.25 > 1.533, we can conclude that the back test #1 of our min-var strategy is statistically significantly less volatile than the S&P/TSX Composite index during this testing period.

Now that we know the two samples have significantly different variance, we can proceed to test the difference of mean using the model for variance inequality. The t stats in this case is calculated to be (0.78%-0.86%)/square root of [(0.08%+0.18%)/113] = - 0.1667. The degree of freedom in this case is equal to 119.58. If we look up the T distribution table in the appendix, obviously we **cannot** conclude that back test #1 produced a statistically different mean monthly return from the S&P/TSX composite index, no matter what confidence level we use.

So we had a "naïve" min-var strategy that didn't work. It was indeed significantly less volatile than the equity index but was unable to beat the index from either a statistical or cumulative perspective. So let us start from scratch to see whether it is possible to engineer a min-var type of portfolio strategy to consistently outperform the market index in Canada as minimum variance literature claims. The first step is to conduct a new back test with a constraint on turnover (maximum 10% a month) so that it is actually realistic to invest in. Back test #2 results are shown in Exhibit 3 and Table 2.

Exhibit 3 Hypothetical Results: Back Test #2

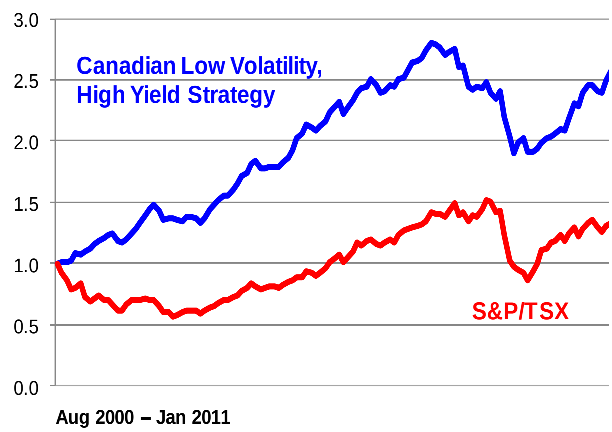


Table 2

Realized Results	**Value**
Annualized Return	10.59%
Average Annualized (Total) Risk	9.71%
Annualized Active Return	**5.99%**
Average Annualized Active Risk	12.23%
Average Period Turnover	**9.92%**
Average Number of Assets	64.5317
Sharpe Ratio	1.0905
Information Ratio	0.4898

The results are encouraging. Excess return "alpha" increased to 5.99% with turnover at barely 10% a month. This means we can retain 4.8% in post-transaction excess performance over the test period (August, 2000 to January, 2011). In addition to improving portfolio performance, realized total portfolio volatility decreased from 9.9% to 9.71% annualized. Theoretically, we can therefore argue that maintaining a true minimum variance strategy (given other artificial portfolio construction constraints) will cause portfolio turnover to be extremely high while transaction costs will eliminate most of the excess performance. Because of the artificial constraints in our back tests, these portfolios are theoretically not at the true "minimum variance" point according to the risk model's estimate. Therefore, going forward, we will refer to our back test strategies as "low volatility" (LV). But, most importantly, how is the relative performance after removing the tech bubble period?

Exhibit 4 Hypothetical Results: Back Test #2

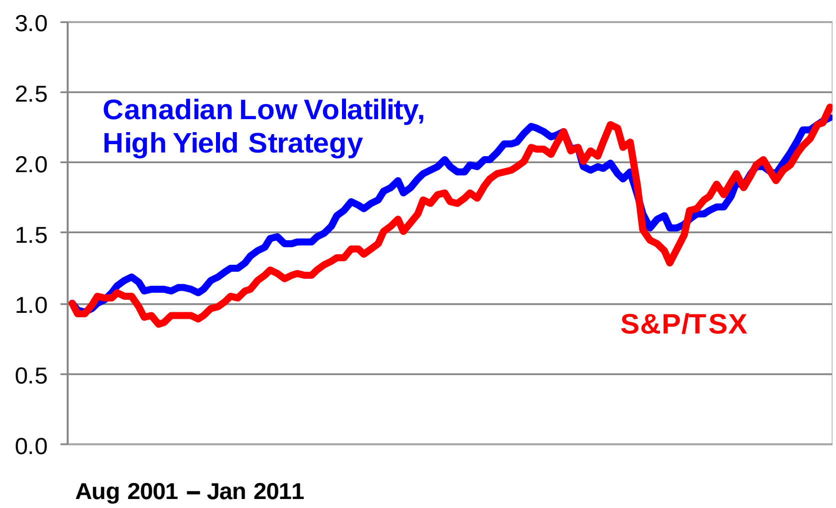


Exhibit 4 starts to look more like it. Even though, this portfolio produced index-like returns over all, at least it was outperforming during market crash and underperformed a bit during the recovery, which is exactly what one would expect a low volatility strategy to do in general.

A number of previous studies have generated similar back test results highlighting the excellent performance of "min-var" strategies over the past decade in Canada. However, going back to some of our questions regarding the technology bubble and the financial crisis period, these claims definitely deserve closer study.

One of the constraints for the back test was a maximum individual stock holding of 4%. This artificial constraint most likely contributed to most of the outperformance when the tech bubble burst; Nortel Networks was over 30% in the index when it blew up quite spectacularly. This certainly was an anomaly in the Canadian market's history and cannot be relied upon to design an out of sample consistent portfolio strategy.

Exhibit 4 shows an interesting performance comparison during the 2007 – 2010 period when the market experienced a significant crash and an amazing recovering. The LV portfolio declined before the index did but outperformed during the low of the market and ended up with returns very similar to the index in early 2011 on a cumulative basis. For the most part, it appears to have done what it was designed to do. However, the fact that it led the market on the downside is a concern. When we looked at the holdings in the portfolio during this period, we noticed that the portfolio had a few concentrated 4% full weightings on consumer discretionary names such as Corus Entertainment and Astral Media. These historically less volatile names led the market on earnings disappointments and significantly underperformed prior to the market downturn. Given how small these stocks are in the S&P/TSX index, 4% weights are quite large active positions in a portfolio and this contributed significantly to the strategy's overall underperformance.

To sum up, our back tests demonstrated the following:

- Over the testing period, the LV portfolio appeared to have some "alpha" at some point in history.
- Short term significant performance can have long-lasting impact on a cumulative basis when returns are geometrically linked (The compounding effect of cumulative returns. Technology bubble period).
- Relative to the market index, the LV portfolio can underperform in history due to the concentration of individual positions in the low volatility names with no additional risk controls on them, even when the strategy is following low risk construction techniques. This calls for better portfolio engineering rather than a simple LV optimization.

For the first point, other research papers have endeavored to explain the return anomaly. For example, the latest research from Nomura securities[2] shows a significant Value and Small cap bias in min-var strategies in the US market, which explains the "alpha". Nomura illustrated this by comparing a min-var portfolio with the Russell 1000 Index and the Russell 2000 Index. They discovered that the "mysterious" alpha only showed up when using the Russell 1000 Index as the benchmark and completely disappeared vs. Russell 2000 Index, which is biased towards small cap names and value. We observed a similar bias in the portfolio created in Back test # 1 (see Table 3 for our LV strategy's average style factor exposure over time in Back Test #1

Table 3

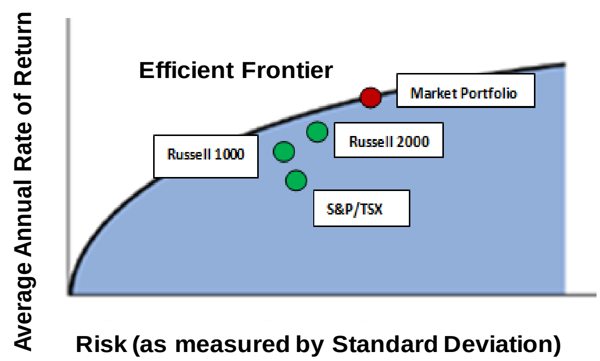
Value	Leverage	Growth	Size	Market sensitivity	Liquidity	Midterm Momentum	Short term momentum	Volatility
0.2729	-0.0610	-0.1032	-1.095	-0.8851	-0.4846	0.1320	-0.1202	-0.6547

We subsequently conducted additional LV strategy back tests and corrected the value and size bias vs. the index by constraining style exposures (In section II). However, we still found cumulative outperformance in the Canadian market. This means that this theory does not explain the anomaly, at least not in all the markets completely. [3] Northfield recently offered a more comprehensive explanation for the anomaly, and explanation with which we tend to agree more. [4]

[2] Joseph J. Mezrich & Yasushi Ishikawa - "Now you see it, now you don't – Low volatility alpha as index distribution arbitrage" July 2011
[3] These additional back tests will be shown and referenced in section II when we compare the different types of min-var portfolios.
[4] Dan diBartolomeo – "A detailed examination of minimum variance and low volatility equity strategies" July 2011

One of their main points (the assumption that the CAPM holds up in a real market environment) is flawed because the equity indices in different markets are not the true "market portfolios" to which CAPM refers. Equity indices are constructed based on arbitrary membership criteria, plus free-float-adjusted market capitalization data, hence the portfolio weighting in the equity indices are never mean-variance efficient. In other words, these index portfolios are not on the Efficient Frontier.

Exhibit 5 Efficient Frontier



Any systematically optimized portfolio (Min-var portfolio, for example) is therefore more efficient on per unit of risk basis, hence the outperformance. Additionally, the more inefficient an equity index, the better the potential for a LV strategy to beat it as a benchmark. (Remember, an LV portfolio optimizes by minimizing the overall risk of the portfolio itself. It does not have any knowledge of the index.) This also explains the difference Nomura Research sees between the Russell 1000 and the Russell 2000 index. The Russell 1000 is indeed dominated by large cap names and is more concentrated. The Russell 2000 is closer to the Efficient Frontier or the true market portfolio. Therefore, it is more difficult to beat. Of course, Canadian benchmark-focused portfolio managers know how concentrated the S&P/TSX index is. It is even further away from the Efficient Frontier and a true market portfolio. Exhibit 5 illustrates this important point.

Obviously, to achieve this, a simple LV strategy in back test #2 would not be sufficient given our observations. We cannot rely on this portfolio to beat the index simply by holding less Nortel because it is too stock-specific to make it a sustainable long term alpha strategy. In addition, the concentration in the consumer discretionary sector and the few small names in this sector have a significant impact on short term performance. The question is: How do we improve this simple strategy to produce more consistent long-term reliable alpha? We seek the answer in Section II.

Section II

The spectrum of different kinds of strategies by enhancing the base case

Even though the simple back tests discussed in Section I are flawed, they provide a great starting point on which to build. In Section II, we would like to introduce a natural evolution process from a simple LV strategy to building a more sophisticated product.

We will start by screening the investment universe and imposing new constraints. Finally, we will change the risk model and also introduce various style constraints to observe if there is any change in performance. We are now able to generate a line-up of LV portfolio strategy back tests: (each new screen is incremental to the previous ones, except for 9, 10 and 11, which are mutually exclusive)

3. Investment universe: Top 200 Canadian stocks and top 80 income trusts in history based on market capitalization, excluding any stocks with dividend less than 1% at each rebalancing date. This is to screen out low yielding stocks.
4. Investment universe: Exclude stocks that yield more than 15%. This is to screen out stocks with extremely high yields (value trap)
5. Investment universe: Exclude stocks that have payout ratio (Dividend/earnings) less than 0 or higher than 100%. This is to avoid dividend cuts on high yield stocks.
6. Investment universe: Exclude stocks that are in the bottom two quintiles (approximately 40%) of Genus model rankings. (These are rankings based on a multi-factor based excess return forecast models consists of value, growth, momentum, quality and sentiment data, the higher the rank the better the probability of a certain stock beats the market with excess returns.)
7. Constraints: Maintain a 4% yield level for the overall portfolio on rebalance dates. This is to add additional return source by keeping the overall yield level.
8. Risk model: Apply Axioma medium horizon **statistical model** (please see Appendix 2) in the back test. This is to determine if a different risk model will produce different results for the back test.
9. Based on back test #8 → Constraints: Maximum size deviation (+/-) from the TSX index to be 0.1 standard deviation using Axioma size factor. (please see Appendix 2 factsheet)
10. Based on back test #8 → Constraints: Maximum value deviation (+/-) from the TSX index to be 0.1 standard deviation using Axioma value factor. (please see Appendix 2 factsheet)
11. Based on back test #8 → Constraints: Maximum momentum deviation (+/-) from the TSX index to be 0.1 standard deviation using Axioma medium term momentum factor. (please see Appendix 2 factsheet)

The summary statistics of these nine new strategies is shown in Table 4.

Table 4									
Realized Results	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
Annualized Return	11.43%	11.35%	12.17%	13.07%	15.59%	15.90%	13.61%	14.21%	15.22%
Annualized (Total) Risk	9.09%	8.98%	9.32%	9.23%	10.14%	9.82%	10.18%	9.71%	10.43%
Annualized Active Return	6.83%	6.76%	7.58%	8.48%	10.99%	11.30%	9.01%	9.61%	10.63%
Annualized Active Risk	12.35%	12.28%	12.44%	12.65%	11.70%	11.72%	10.67%	11.35%	11.13%
Average Period Turnover	19.84%	19.84%	19.80%	19.84%	19.35%	19.44%	20.59%	19.86%	19.50%
Average Number of Assets	52.373	51.9683	41.7063	40.4048	39.0397	39.0397	39.0873	39.0397	39.0635
Sharpe Ratio	1.2571	1.2636	1.3068	1.4169	1.5381	1.6194	1.3374	1.4639	1.4598
Information Ratio	0.5532	0.5502	0.6092	0.6698	0.9394	0.9644	0.8449	0.8467	0.9544

Exhibit 6 shows the performance comparison of the nine new strategies vs. the first two we did in section I.

Exhibit 6 Hypothetical Results: Back Tests #3-11

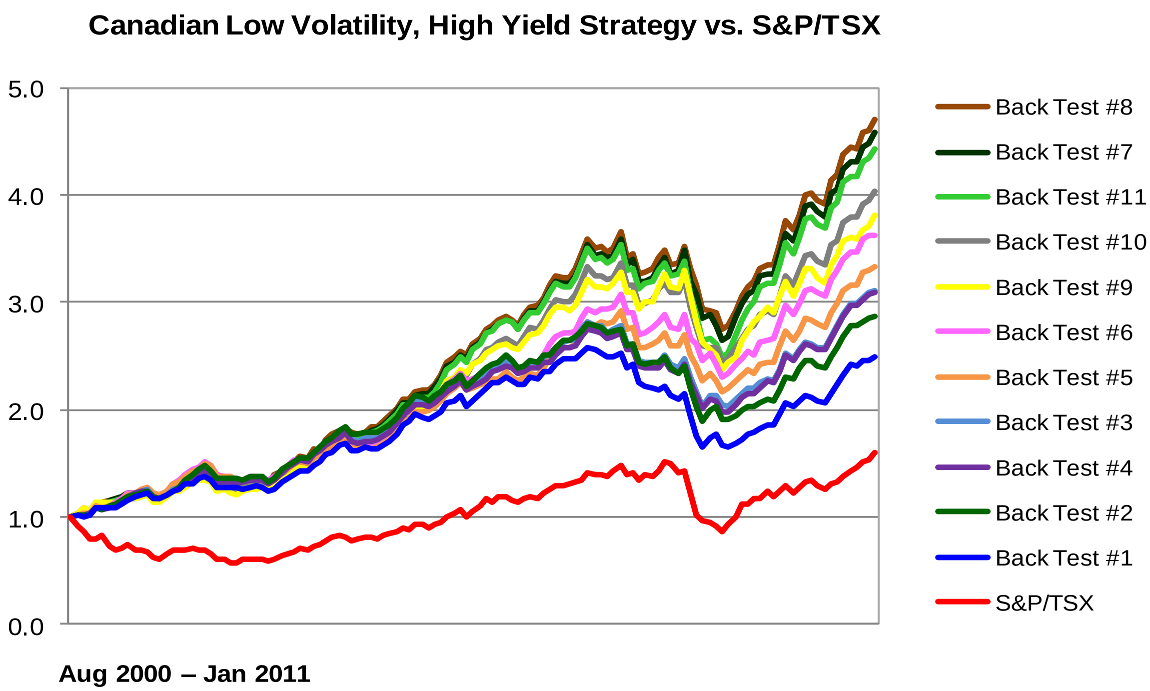
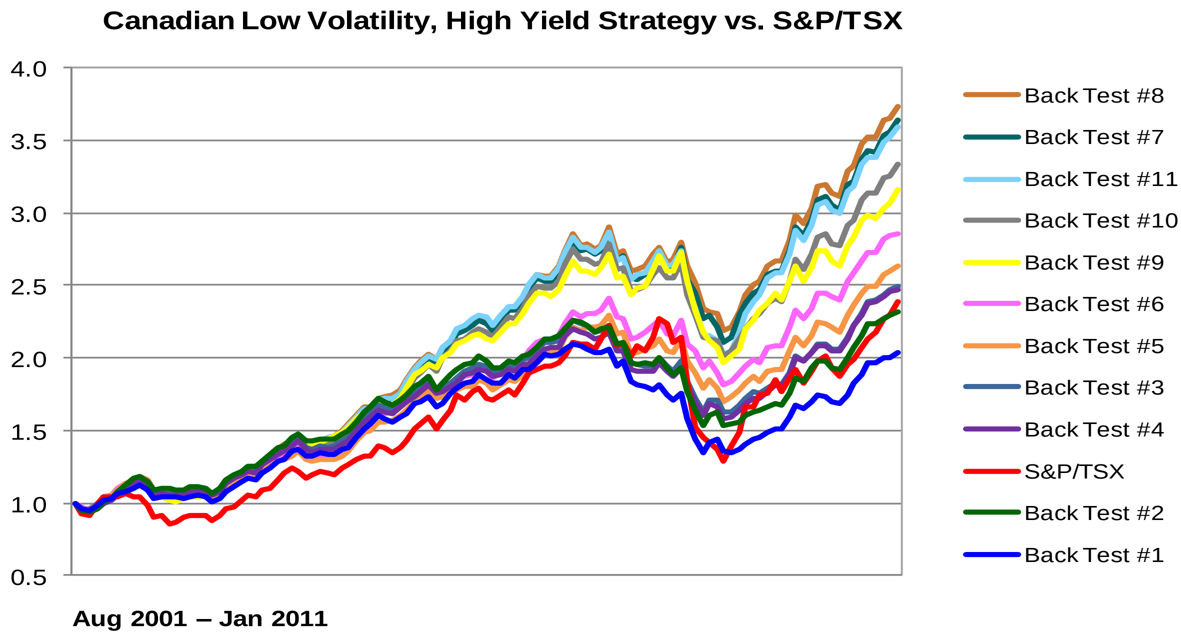


Exhibit 7 shows the results if we start our back tests in August 2001.

Exhibit 7	Hypothetical Results: Back Tests #3-11



Total return, total risk and Sharpe ratios for the above back tests are presented in Table 5.

Table 5

Back Test	3	4	5	6	7	8	9	10	11	TSX
Total Annualized Return	10.07%	10.00%	10.72%	11.66%	14.57%	14.89%	12.87%	13.50%	14.40%	9.61%
Total Annualized Risk	9.29%	9.18%	9.54%	9.38%	10.35%	9.97%	10.25%	9.97%	10.68%	14.80%
Sharpe Ratio	1.083	1.089	1.124	1.243	10.407	1.494	1.255	1.355	1.348	0.649

These portfolios now are separated into two sub groups based on their performance and behavior: back test #3, #4, #5, and #6 (Group 1) vs. back test #7, #8, #9, #10 and #11 (Group 2). One significant difference in designing these two groups is the constraint to keep overall yield level high at 4%. This turned out to be a key condition for Group 2 to outperform Group 1. Most research to date into low volatility strategies has argued that a relatively higher yield will be a by-product of a low volatility strategy. We, on the other hand, emphasize that a sustainable yield component is essential for designing a superior low volatility strategy in Canada. This is achieved by explicitly constraining the overall yield level of the strategy. Therefore we can now refer to this as a low volatility, high yield (LVHY) strategy.

In fact, assuming a 1.2% linear transaction cost as we did in Section I, none of the back tests in Group 1 makes realistic sense as an investment product because the alphas mostly disappear once we deduct the 1.2% transaction cost. (#6 will have a little residual: 11.66% - 9.61% - 1.2% = 0.85% per annum, but management and other fees will take away that too). Therefore, we need excess returns to be much larger to make a truly viable investable product.

Looking at Group 1, we see that back tests #3, #4 and #5 all underperformed the TSX, even before the market collapsed in 2008. Only #6 outperformed the market index throughout the financial crisis and the subsequent recovery. However, back test #6 explicitly screened out the bottom two quintiles using the Genus quantitative stock selection models. These models drove more high quality fundamentals into the portfolio construction process. (The Genus models focus on wealth-generating factors such as earnings and cash flow oriented valuations, strong momentum, as well as expectational factors and measures of financial quality such as balance sheet strength and ROE, etc.) This positively impacted the performance of back test #6 vs. all the other back tests in Group 1 and also established a strong foundation for the superior back tests in Group 2.

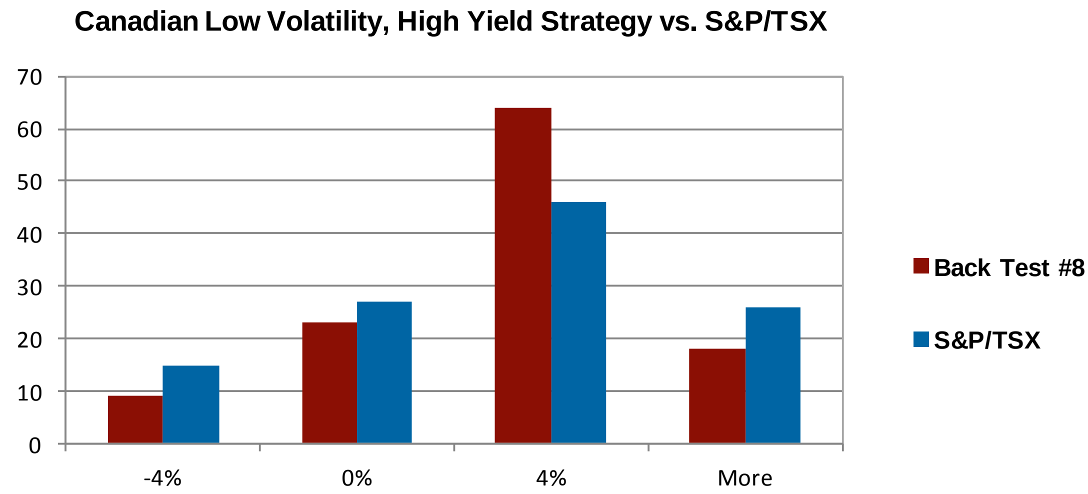
Within Group 2, back test #8 offers the best performance and Sharpe ratio statistics. Back tests #7 and #8 are exactly the same strategies, except with different risk models. Back test #8 uses the Axioma statistical model while #7 uses the fundamental model. The results show that the statistical model provides better risk-adjusted performance. Axioma has researched these two models and concludes that the statistical model seems to capture total risk more accurately when the market is extremely volatile, and vice versa for the less volatile periods in history (The explanation behind this was that during market turmoil, all fundamental risk factors such as value, growth and size, etc. start to have higher than normal correlations, therefore, the explanation power or predictability of this type of risk models could really drop. While on the other hand, a Principle Component Analysis based statistical risk model only aims at breaking down risk exposures from mathematics point of view, which proven more effective in capturing the unusual risk contributions). Given that it is the best portfolio in these back tests; let us drill down to provide more insights into its historical performance.

Table 6

Realized Results	Back Test #8	S&P/TSX
Best 1 Month	6.34%	11.46%
Worst 1 Month	-7.13%	-16.75%
Best 1 Year	43.45%	48.36%
Worst 1 Year	-16.02%	-38.12%
Maximum Drawdown	-23.98%	-53.36%
Negative Mean	-2.38%	-4.44%
Negative Standard Deviation	2.16%	3.54%
Negative Median	-1.67%	-3.64%
Positive Mean	2.60%	2.93%
Positive Standard Deviation	1.65%	2.29%
Positive Median	2.45%	3.05%

We observed that back test #8 offers more stable returns with both the up and the down side limited to around 7% per month. And on an annualized basis, it appears to capture most of the upside and largely reduces the down side (rolling 1 year max and min). Additionally, the volatility of #8's negative performance is measurably smaller vs. the TSX. The realized correlation of monthly performance with TSX index is 0.79, which is very close to the average predicted beta in the back test. Exhibit 8 shows the monthly performance comparison of return frequencies in different ranges:

Exhibit 8 Hypothetical Results: Back Test #8 Histogram



Aug 2001 – Jan 2011

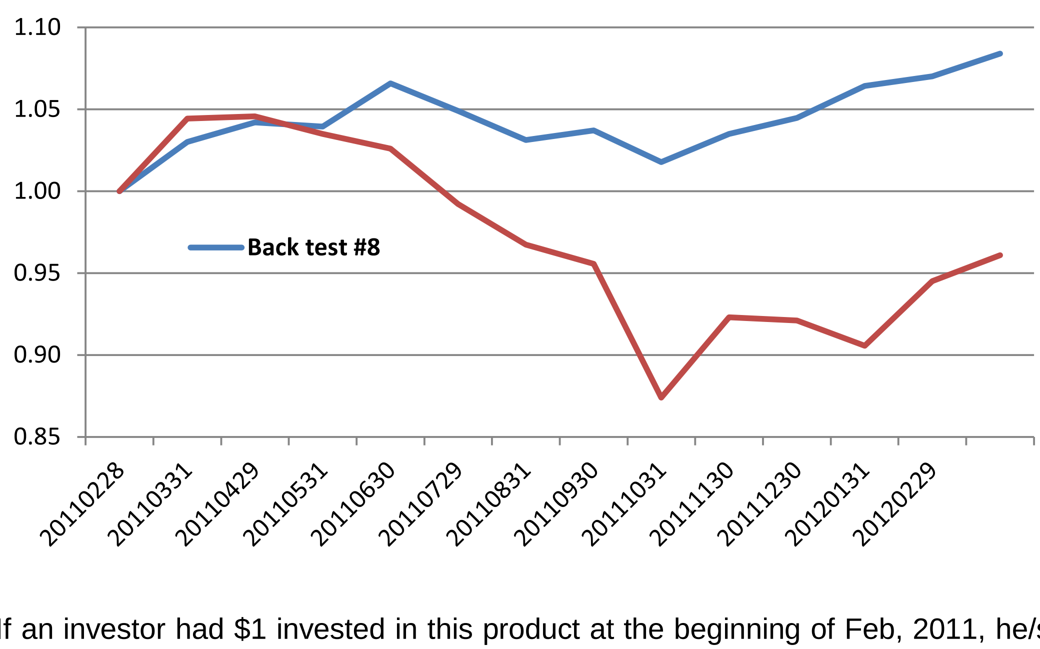
Cleary, back test #8 provides downside protection by giving up some returns on the upside. But, more importantly, it has a larger portion of monthly returns in the 4% range vs. the TSX index. This highlights quite clearly the essence of the LV strategy: ***more consistent returns.***

Now the moment of truth: let us refer to our statistical test to see whether our best strategy (back test #8) has a statistically different variance and mean return vs. the S&P/TSX Composite Index for the in-sample period of Aug 2001 to Jan 2011. Following the footsteps of conducting this test for back test #1 back in section I, we could ascertain that the F statistics = 0.18%/0.08% = 2.25. With the same degree of freedom, we can conclude that back test # 8 is also significantly less volatile than the Index. The mean monthly return from this best strategy was 1.20%. So using the t test for the case of variance inequality, the t stats with the same degree of freedom turned out to be: (1.20%-0.86%)/square root of [(0.08%+0.18%)/113] = 0.7089. Looking up the t distribution table, we can see that unless we use relatively lower confidence level of 75% or less, the mean monthly return of this best strategy is **NOT** statistically different from the index. We will conclude, however, that with a significantly lower volatility and statistically index-like returns, this best strategy is significantly superior to the S&P/TSX Composite Index from a risk-adjusted (Sharp ratio) perspective and the enhancements did improve the "naïve" strategy, which actually underperformed the index.

Back tests #9, #10 and #11 are designed to look at the impact of style bias in these LV strategies. Recall in Section I we stated that despite adjusting for style biases such as size and value, there is still cumulative outperformance to be obtained from LV portfolios in Canada vs. the S&P/TSX index. We can now examine these three back tests, we limited the net exposure of size, value and momentum to remove the style factor bias against the index. Theoretically, we do not know if these results would be risk model specific, however, these common risk factors as predominate style factors have been researched by risk model providers and alpha explorers so much that these definitions have a very high degree of correlation across many vendors and practitioners. So using an alternative model or definition will most likely yield different figures but I would highly doubt that will change the conclusions.) The results still generated cumulative alpha vs. the TSX (approximately 3% to 5% before fees and transaction costs). One interesting observation is that the size constraint appears to have had the most impact on overall performance, followed by value and then momentum. Recall that Table 2 shows a low volatility strategy will have the following features: Small cap, good value, low beta and high momentum with size being the most significant bias vs. the S&P/TSX index.

All the research so far has focused on in-sample strategy performances; it is now time to study out-of-sample results. Amazingly, during the out-of-sample period of Feb, 2011 to Feb, 2012, the market has endured another “mini-cycle” featuring the unfolding of the European debt crisis, the recovery of the US economy and a government-engineered slowdown of China’s red-hot economic growth. The US equity market experienced more than 20% in draw downs in the summer of 2011, followed by a quick recovery of more than 15% in the fall, another 10% correction before the holiday season, and a sustained rally of more than 20% since then. It looks as if the only reassuring aspect is the volatility. So, how did our best Canadian strategies fare during this “out-of-sample” period compared to the index? Exhibit 11 (a) shows the comparable cumulative total return.

Exhibit 11 (a) Hypothetical Results – Back Test #8 Out-of-sample Cumulative Total Returns



If an investor had $1 invested in this product at the beginning of Feb, 2011, he/she would have not lost money at any point during the following year. In fact, they would have ended up making more than 8 cents cumulatively. On the other hand, an index fund would have lost more than 12 cents for the investor after the summer of 2011, but would have recouped some losses to end the year for a total loss of 4 cents.

To put the out-of-sample results in the same perspective as our previous discussions, Exhibit 11 (b) shows the monthly performance of back test number #8 vs. the S&P TSX Composite index, and Exhibit 11 (c) the detailed performance analysis:

Exhibit 11 (b) Hypothetical Results – Back Test #8 Canada, Out-of-sample Histogram

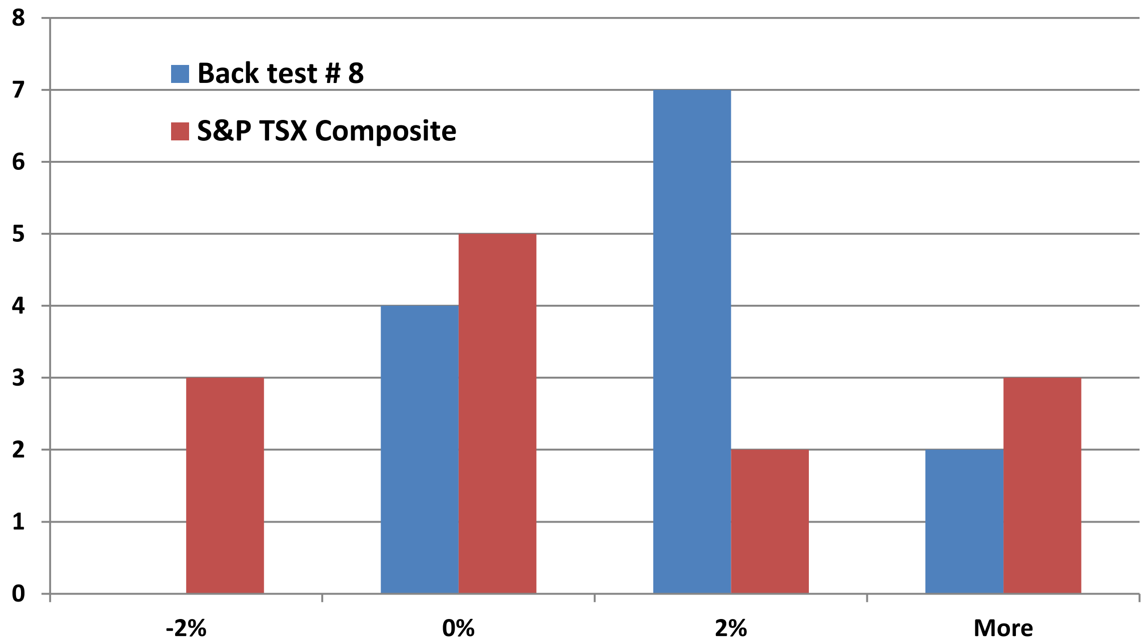


Exhibit 11 (c)

	Back test # 8	S&P TSX Composite
Cumulative return	8.40%	-3.91%
Annualized risk	5.49%	12.99%
Sharp Ratio	1.53	-0.30
Mean Monthly	0.63%	-0.24%
Monthly volatility	1.58%	3.75%
Best 1 month	3.00%	5.62%
Worst 1 month	-1.88%	-8.54%
Max draw down	-4.57%	-17.44%
Positive mean	1.52%	3.24%
Negative mean	-1.35%	-2.42%
Linear Correlation (beta)	***0.69***	

Statistically, F statistic revealed a significant different volatility of this strategy out-of-sample as well vs. the TSX. (F = 6.5). And the mean return t stat is calculated to be 0.90 with a degree of freedom of 15.2. Therefore, it has significantly higher return than the index at 80% confidence level, a little better than the in-sample result. From the data and chart, we can comfortably conclude that our strategy behaved exactly as we designed it to: consistent, low risk, good cumulative outperformance. Notice that the "Canadian best strategy" did not have any months with returns lower than -2%, and the majority of returns are between 0% and 2%. The index monthly returns are a skewed towards the two tails in the histogram. Surprisingly, the out-of-sample sharp ratio is also very close to the in-sample back test results: 1.53 vs. 1.49.

In summary, we learned the following from our LV strategy back tests in the Canadian market:

- Over time, low volatility stocks behave very differently vs. the overall market but statistically only from a volatility stand point, not really for monthly returns (at least not with more than 75% confidence).
- A simple min-var strategy creates significant turnover that takes away most of the "alpha". Therefore, a turnover constrained approach is a better way to start.
- The LV strategy can be enhanced and significantly improved by adding quality screens on the universe.
- Introducing a yield component can have a material impact on the overall performance of the LV strategy.
- A statistical risk model produced better risk-adjusted performance vs. traditional fundamental-based risk model.
- Any low volatility strategy will have a small-cap value bias vs. the general market index. The small cap effect is by far the largest bias. But even removing these style biases, LVHY strategies are still able to outperform the S&P/TSX composite index, which is an arbitrarily created inefficient portfolio.
- This evolution of LV strategies showed us that there are actually many methods and features an LV-oriented investment strategy can have. Careful theoretical consideration and empirical studies are required to obtain a superior investable product.

Section III

A global setting

After exploring LV strategies in Canada, we expanded our investment universe to the global equity markets to determine if similar strategies and empirical results could be produced. In this regard, the MSCI World Index is our reference market index for performance comparisons.

We started with our base-case back test #1's environment and parameters:

- Back test period: Feb 1999 to May 2011 (Tech bubble, commodity boom, financial crisis and quantitative easing).
- Market: Global developed equity markets
- Reference benchmark: MSCI World Total Return Index
- Risk model: Axioma Worldwide Fundamental Risk Model
- Investment Universe:
  - Canada: Top 200 Canadian stocks and top 80 income trusts based on market capitalization
  - US: Top 900 stocks based on free-float capitalization in Russell 1000 index and S&P 500 index.
  - EAFE: Top 2000 stocks based on free-float capitalization in Russell global index and MSCI EAFE Index.
- Currency: USD
- Objective: Minimize the overall risk of the designed portfolio, rebalancing monthly.
  Subject to the following constraints:
  GICS sectors weighting: Max 25% each
  Income trust holdings: Max 15%
  Individual income trust names: Max 1% each
  (These two constraints are to mitigate the historical "income trust" effect)
  No country constraints
  Range for number of holdings: 40 to 80
  Maximum individual stock weights: 2.5%
  Turnover constraints: None
  Transaction cost: None.

Then, keeping everything the same, we imposed all the incremental constraints and parameters to generate back tests 2 to11 using the same methodology and process followed in Section II:

1. Unconstrained on turnover, minimum variance strategy.
2. LV strategy with turnover constrained at Max 10% per month.
3. Screen out low yielding stocks.
4. Screen out extremely high yielding stocks.
5. Screen out stocks with unsustainable dividend payouts.
6. Screen out low ranked stocks based on Genus' quantitative stock selection models.
7. Impose a minimum portfolio yield constraint (LVHY strategy).
8. Use Axioma Statistic Risk Model.
9. Limit overall portfolio size exposure relative to the reference benchmark.
10. Limit overall portfolio value exposure relative to the reference benchmark.
11. Limit overall portfolio momentum exposure relative to the reference benchmark.

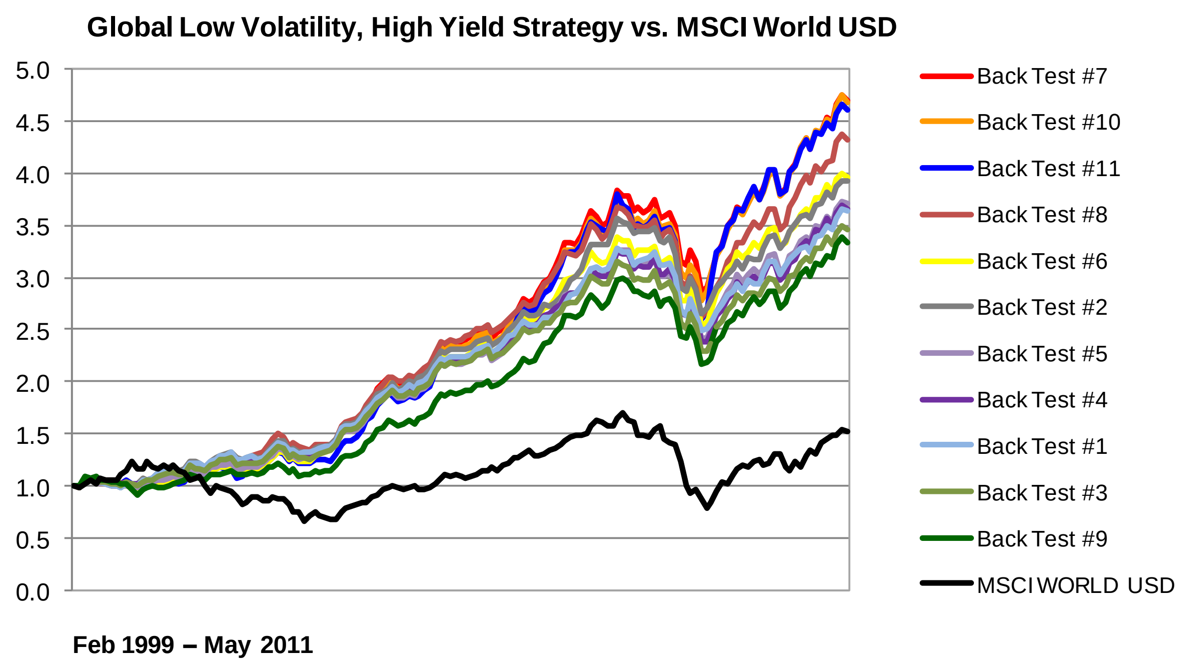
Table 7 and Exhibit 9 show the subsequent results and statistics.

Table 7

Realized Results	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
Annualized Return	10.74%	11.41%	10.32%	10.83%	10.92%	11.51%	13.03%	12.29%	9.93%	12.99%	12.92%
Average Annualized (Total) Risk	9.01%	9.08%	9.60%	9.64%	9.40%	9.57%	10.09%	10.14%	10.27%	10.05%	11.75%
Reference Benchmark Return	3.44%	3.44%	3.44%	3.44%	3.44%	3.44%	3.44%	3.44%	3.44%	3.44%	3.44%
Annualized Active Return	7.30%	7.97%	6.88%	7.39%	7.48%	8.07%	9.59%	8.85%	6.49%	9.55%	9.48%
Average Annualized Active Risk	16.97%	16.78%	16.89%	16.87%	17.13%	17.18%	17.28%	17.24%	17.66%	17.29%	17.63%
Average Period Turnover	34.00%	9.98%	20.28%	10.52%	10.90%	11.23%	11.49%	11.46%	11.19%	11.46%	11.48%
Average number of Assets per period	76	78	76	76	74	73	71	68	62	71	68
Sharpe Ratio	1.193	1.258	1.075	1.124	1.162	1.203	1.292	1.212	0.967	1.292	1.100
Information Ratio	0.430	0.475	0.407	0.438	0.437	0.470	0.555	0.513	0.368	0.552	0.538

Note: MSCI World Index annualized return during the testing period was 3.4% in USD terms. Realized annual volatility was 16.47%. Sharpe Ratio of 0.21.

Exhibit 9 Hypothetical Results: Back Tests #1-11



Examining the back test results in the global market, the following observations are worth mentioning (some of them are quite different vs. the Canadian market back tests in the previous section):

- Constraining turnover produced higher return and lower realized volatility than min-var without a turnover constraint.
- Screening out low, high and unsustainable yielding stocks did not add much value on their own. (Back tests #3, #4, #5 all had lower return than #2). Once we screened further based on the Genus rankings, we were able to obtain a new high in terms of returns and also Sharpe ratio.
- The biggest improvement on return and risk adjusted return still came from the minimum yield constraint in #7, which shows the attractiveness of LVHY, i.e. designing the yield component separately instead of treating it as a LV by-product.
- Interestingly, the statistical model did not generate a superior product vs. its fundamental counterpart, in contrast to the Canadian tests. This suggests that the global equity market might be better analyzed from a fundamental perspective. Of course, more detailed analysis is required to make that assertion, which goes beyond the scope of this study. [5]
- The marginal impact of size bias on performance is even bigger in the global setting.

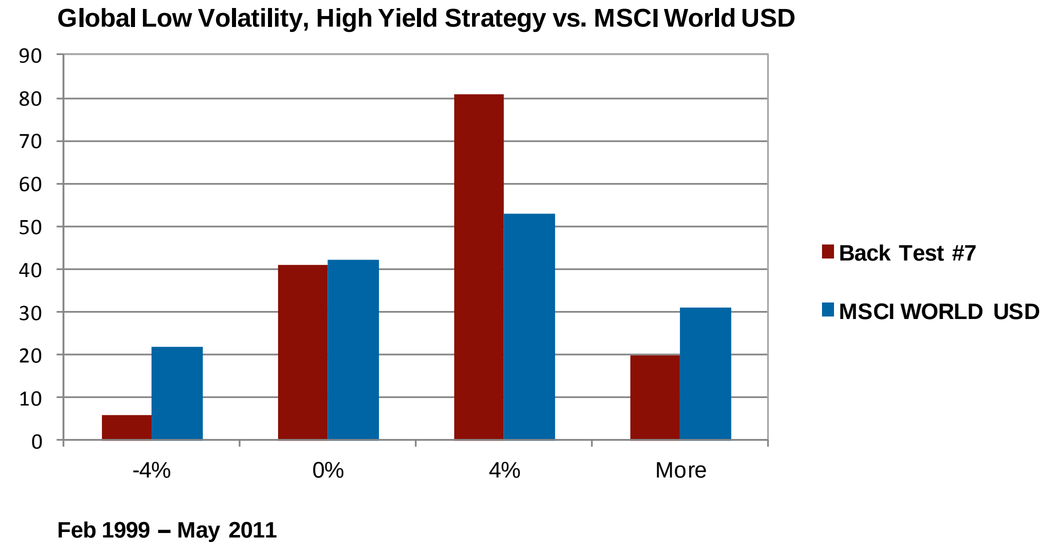
[5] Robert Stubbs, VP of Axioma research has a detailed study on this topic - " Advantages of Multiple Risk Models in Portfolio Management" 2009

- Value and momentum bias did not seem to matter much in terms of performance. But limiting momentum exposure increased realized volatility significantly. In fact, back test #11 generated the most volatile portfolio. From a quantitative research perspective, we know that momentum stocks performed well relative to the general market during the market crash in 2008, but lagged the recovery in 2009 and 2010. Therefore, constraining the exposure to momentum will make this strategy underperform during a market downturn (more downside vs. unconstrained LVHY) and outperform during a market recovery (more upside vs. unconstrained LVHY), hence the higher realized volatility. Exhibit 9 shows exactly this. Notice how back test #11 (blue line) underperformed the best strategy (back test #7) and then outperformed it through the 2008 and 2009 period.
- We conclude Section III by comparing performance statistics vs. the reference benchmark. Table 8 shows they look very similar to the Canadian statistics, except for the realized beta (0.22).

Table 8

Statistics	Global LVHY (#7)	MSCI World
Best 1 Month	7.58%	11.32%
Worst 1 Month	-9.70%	-18.93%
Best 1 Year	50.51%	55.18%
Worst 1 Year	-22.02%	-46.76%
Maximum Drawdown	-27.73%	-64.55%
Negative Mean	-2.18%	-4.96%
Negative Standard Deviation	2.09%	3.50%
Negative Median	-1.42%	-3.64%
Positive Mean	2.62%	2.89%
Positive Standard Deviation	1.74%	2.81%
Positive Median	2.25%	2.28%

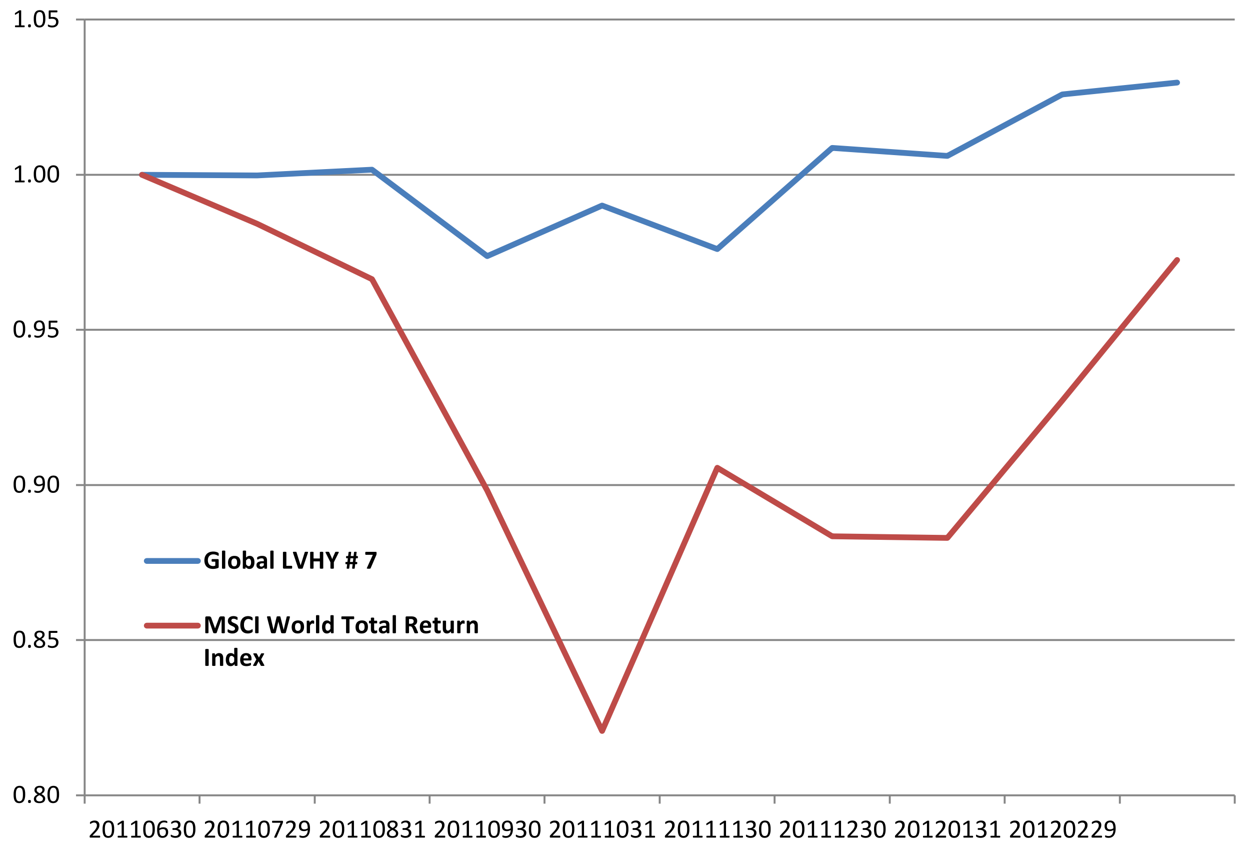
Exhibit 10 Hypothetical Results: Back Test #7 Histogram



Of course, after closely examining the performance data for the best global strategy, we will conduct the difference of mean and variance tests to see if we can unveil anything different. Please note that the global strategy features a longer in-sample period of Feb, 1999 to May, 2011. F = 0.23%/0.08% = 2.875 with degree of freedom of 147. So apparently, the best global strategy also is significantly less risky than the MSCI World total return Index. T stats of the difference of mean return equals to 1.52 with a degree of freedom of 244.75. Now, we can conclude that the global best LVHY strategy is not only significantly less volatile than the MSCI World Index; it also amazingly has higher return with approximately 93% confidence level!

To complete our research, we also conducted the out-of-sample testing in our Global setting. Exhibits 12a, 12b and 12c show the performance chart, histogram and the detailed analysis during the Global strategies' out-of-sample period, which is June 2011 to Feb, 2012.

Exhibit 12 (a) Hypothetical Results – Back Test #8 Global, Out-of-sample Histogram



As Exhibit 13 (a) shows, the global market correction and rally was very similar to the experience in Canada, noting that the time period starts in June 2011, which is right before all the volatility kicked in. This explains why the index return during the testing period seemed a bit more drastic.

Exhibit 12 (b)

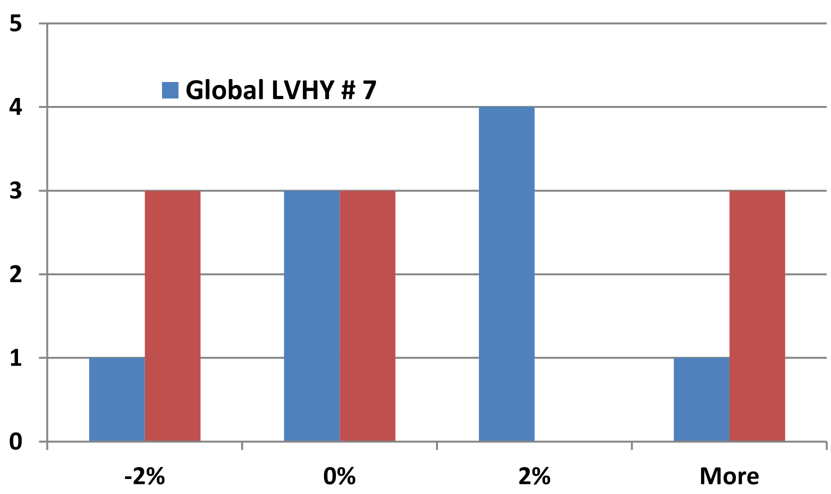


Exhibit 12 (c)

	Global LVHY # 7	MSCI World Total Return Index
Total Return	2.97%	-2.74%
Annualized risk	N/A (Less than 12 months data to calculate)	
Mean Return	0.34%	-0.15%
Monthly Risk	1.83%	6.04%
Best 1 month	3.35%	10.34%
Worst 1 month	-2.79%	-8.64%
Max draw down	-2.79%	-19.08%
Positive mean	1.51%	6.75%
Negative mean	-1.12%	-3.60%
Correlation (Beta)	***-0.07***	

For the out-of-sample data, the statistical conclusion changes back to the same as the Canadian best strategy: significantly less volatile but not statistically higher return than the reference index. (Out sample period was however very short with 9 months of observations only)

Section IV

Conclusion

Through our empirical back test study in both the Canadian and the global equity markets, we have learned that the return anomaly of low volatility portfolios does exist from a cumulative excess return perspective. However, an investable strategy is not easily obtained by simply minimizing risk. There are more enhancements and additional portfolio designs that are required to produce long term consistent absolute and relative risk-adjusted returns.

As a conclusion, we would suggest the settings on back test #8 in Section II and back test #7 in Section III are best-suited for a Canadian and a Global LVHY product respectively. The best global LVHY strategy (#7) also exhibits very similar behavior to its Canadian counterpart. Even though the testing period was a little different vs. the Canadian out-of-sample period, the return distribution and statistics all point to the same conclusions: the best strategy in the global setting also performed according to design. One interesting feature do stand out: recall that in our global back testing section, we noted that the realized beta of the best global LVHY strategy was lower than the Canadian best strategy, (0.22 vs. 0.79), relative to their respective reference benchmarks. And for out-of-sample period, the best global strategy had a realized beta of -0.07 vs. MSCI world total return index, almost uncorrelated (Canadian strategy is 0.69 as shown in the table). This beta phenomenon of the Canadian and Global LVHY strategies is also very consistent between in-sample back testing and out-of-sample results. This is in fact re-enforced by our statistical difference of mean tests. Even though both the Canadian and Global strategy showed statistically significance on their low volatility feature, only the Global strategy In-sample return series achieved statistically significant “alpha” vs. the reference benchmark. (We do believe that this makes intuitive sense given the “macro” theme of the Canadian equity market that is concentrated in resource and financial sectors with overall higher beta). This suggests that a global low volatility high yield strategy might be preferred if an investor is seeking to achieve both low volatility and high alpha at the same time and it is the best candidate to become a true “alternative” equity product with pure equity contents!

To conclude this paper, we would like to address a few more discussion points and questions that deserve more on-going research:

- One variable we have not studied in depth in this paper is beta. There are many ways to compute the beta of a portfolio. But do the betas of stocks accurately reflect their risk premiums according to CAPM? Obviously not in the case of the low volatility stocks (Security Market Line is not exactly steep).

- Will the low volatility trade get crowded? How do you monitor it? (We currently have a mechanism designed to study this on an on-going basis.)
- What is the impact of using different risk models, theoretically and operationally? Proprietary risk models or commercial? What about tail-risk modeling and optimization with that? (Notice that even the best strategy had a fairly significant draw down during the worst of the market riot in 2008, which means it is not yet exactly “bullet-proof”)

Designing a superior investment product is a never-ending process and one cannot cease to improve and search for answers to new questions. Hopefully, this paper sheds meaningful light on minimum variance or low volatility strategies and helps investors gain a deeper understanding and insight into these increasingly popular types of investment products.

Appendix 1

F - Distribution ($\alpha = 0.01$ in the Right Tail)

df_2 \ df_1	10	12	15	20	24	30	40	60	120	∞
1	6055.8	6106.3	6157.3	6208.7	6234.6	6260.6	6286.8	6313.0	6339.4	6365.9
2	99.399	99.416	99.433	99.449	99.458	99.466	99.474	99.482	99.491	99.499
3	27.229	27.052	26.872	26.690	26.598	26.505	26.411	26.316	26.221	26.125
4	14.546	14.374	14.198	14.020	13.929	13.838	13.745	13.652	13.558	13.463
5	10.051	9.8883	9.7222	9.5526	9.4665	9.3793	9.2912	9.2020	9.1118	9.0204
6	7.8741	7.7183	7.5590	7.3958	7.3127	7.2285	7.1432	7.0567	6.9690	6.8800
7	6.6201	6.4691	6.3143	6.1554	6.0743	5.9920	5.9084	5.8236	5.7373	5.6495
8	5.8143	5.6667	5.5151	5.3591	5.2793	5.1981	5.1156	5.0316	4.9461	4.8588
9	5.2565	5.1114	4.9621	4.8080	4.7290	4.6486	4.5666	4.4831	4.3978	4.3105
10	4.8491	4.7059	4.5581	4.4054	4.3269	4.2469	4.1653	4.0819	3.9965	3.9090
11	4.5393	4.3974	4.2509	4.0990	4.0209	3.9411	3.8596	3.7761	3.6904	3.6024
12	4.2961	4.1553	4.0096	3.8584	3.7805	3.7008	3.6192	3.5355	3.4494	3.3608
13	4.1003	4.9603	3.8154	3.6646	3.5868	3.5070	3.4253	3.3413	3.2548	3.1654
14	3.9394	3.8001	3.6557	3.5052	3.4274	3.3476	3.2656	3.1813	3.0942	3.0040
15	3.8049	3.6662	3.5222	3.3719	3.2940	3.2141	3.1319	3.0471	2.9595	2.8684
16	3.6909	3.5527	3.4089	3.2587	3.1808	3.1007	3.0182	2.9330	2.8447	2.7528
17	3.5931	3.4552	3.3117	3.1615	3.0835	3.0032	2.9205	2.8348	2.7459	2.6530
18	3.5082	3.3706	3.2273	3.0771	2.9990	2.9185	2.8354	2.7493	2.6597	2.5660
19	3.4338	3.2965	3.1533	3.0031	2.9249	2.8442	2.7608	2.6742	2.5839	2.4893
20	3.3682	3.2311	3.0880	2.9377	2.8594	2.7785	2.6947	2.6077	2.5168	2.4212
21	3.3098	3.1730	3.0300	2.8796	2.8010	2.7200	2.6359	2.5484	2.4568	2.3603
22	3.2576	3.1209	2.9779	2.8274	2.7488	2.6675	2.5831	2.4951	2.4029	2.3055
23	3.2106	3.0740	2.9311	2.7805	2.7017	2.6202	2.5355	2.4471	2.3542	2.2558
24	3.1681	3.0316	2.8887	2.7380	2.6591	2.5773	2.4923	2.4035	2.3100	2.2107
25	3.1294	2.9931	2.8502	2.6993	2.6203	2.5383	2.4530	2.3637	2.2696	2.1694
26	3.0941	2.9578	2.8150	2.6640	2.5848	2.5026	2.4170	2.3273	2.2325	2.1315
27	3.0618	2.9256	2.7827	2.6316	2.5522	2.4699	2.3840	2.2938	2.1985	2.0965
28	3.0320	2.8959	2.7530	2.6017	2.5223	2.4397	2.3535	2.2629	2.1670	2.0642
29	3.0045	2.8685	2.7256	2.5742	2.4946	2.4118	2.3253	2.2344	2.1379	2.0342
30	2.9791	2.8431	2.7002	2.5487	2.4689	2.3860	2.2992	2.2079	2.1108	2.0062
40	2.8005	2.6648	2.5216	2.3689	2.2880	2.2034	2.1142	2.0194	1.9172	1.8047
60	2.6318	2.4961	2.3523	2.1978	2.1154	2.0285	1.9360	1.8363	1.7263	1.6006
120	2.4721	2.3363	2.1915	2.0346	1.9500	1.8600	1.7628	1.6557	1.5330	1.3805
∞	2.3209	2.1847	2.0385	1.8783	1.7908	1.6964	1.5923	1.4730	1.3246	1.0000

Numerator Degrees of Freedom (df_1); Denominator Degrees of Freedom (df_2)

TABLE B: *t*-DISTRIBUTION CRITICAL VALUES

	Tail probability p											
df	.25	.20	.15	.10	.05	.025	.02	.01	.005	.0025	.001	.0005
1	1.000	1.376	1.963	3.078	6.314	12.71	15.89	31.82	63.66	127.3	318.3	636.6
2	.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.09	22.33	31.60
3	.765	.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.21	12.92
4	.741	.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	.727	.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893	6.869
6	.718	.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	.711	.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	.706	.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	.703	.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	.700	.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	.697	.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	.695	.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930	4.318
13	.694	.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	.692	.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	.691	.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	.690	.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	.689	.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	.688	.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.611	3.922
19	.688	.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	.687	.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552	3.850
21	.686	.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	.686	.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	.685	.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	.685	.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	.684	.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	.684	.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	.684	.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421	3.690
28	.683	.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	.683	.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396	3.659
30	.683	.854	1.055	1.310	1.697	2.042	2.147	2.457	2.750	3.030	3.385	3.646
40	.681	.851	1.050	1.303	1.684	2.021	2.123	2.423	2.704	2.971	3.307	3.551
50	.679	.849	1.047	1.299	1.676	2.009	2.109	2.403	2.678	2.937	3.261	3.496
60	.679	.848	1.045	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.232	3.460
80	.678	.846	1.043	1.292	1.664	1.990	2.088	2.374	2.639	2.887	3.195	3.416
100	.677	.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.174	3.390
1000	.675	.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581	2.813	3.098	3.300
∞	.674	.841	1.036	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.091	3.291
	50%	60%	70%	80%	90%	95%	96%	98%	99%	99.5%	99.8%	99.9%
	Confidence level C											

Period	Back test #1	Back test #8	TSX
20010928	-4.74%	-6.24%	-7.37%
20011031	-1.44%	0.76%	-0.77%
20011130	2.79%	4.61%	7.95%
20011231	4.23%	5.11%	5.26%
20020131	2.28%	1.76%	-0.43%
20020228	4.64%	2.41%	-0.10%
20020328	4.54%	3.69%	3.03%
20020430	3.52%	0.27%	-2.39%
20020531	1.85%	1.38%	0.05%
20020628	-2.91%	-1.76%	-6.35%
20020731	-5.39%	-5.30%	-7.45%
20020830	0.97%	-0.76%	0.20%
20020930	-0.02%	-0.37%	-6.24%
20021031	-0.59%	-1.50%	1.21%
20021129	-0.93%	2.25%	5.29%
20021231	2.25%	1.32%	0.93%
20030131	0.26%	0.05%	-0.59%
20030228	-0.51%	-0.76%	-0.02%
20030331	-3.21%	-2.11%	-2.95%
20030430	2.53%	6.27%	3.92%
20030530	5.76%	2.21%	4.31%
20030630	2.66%	3.06%	2.08%
20030731	2.34%	3.00%	3.98%
20030829	2.53%	3.61%	3.43%
20030930	0.09%	-0.85%	-0.97%
20031031	3.32%	4.85%	4.84%
20031128	3.23%	-0.13%	1.28%
20031231	3.21%	5.92%	4.86%
20040130	1.51%	2.55%	3.75%
20040227	4.12%	1.86%	3.24%
20040331	1.57%	1.48%	-2.09%
20040430	-3.43%	-1.63%	-3.74%
20040531	-0.11%	-0.87%	2.28%
20040630	1.00%	0.95%	1.70%
20040730	-0.29%	2.02%	-0.96%
20040831	0.11%	0.87%	-0.81%
20040930	2.53%	1.91%	3.64%
20041029	1.52%	3.18%	2.44%
20041130	3.73%	3.88%	1.94%
20041231	4.79%	3.88%	2.66%
20050131	2.21%	-0.14%	-0.64%
20050228	3.46%	3.42%	5.17%
20050331	-1.33%	0.87%	-0.31%
20050429	-1.26%	0.22%	-2.41%
20050531	2.15%	2.18%	2.70%
20050630	1.63%	3.99%	3.07%

20050729	3.44%	4.73%	5.33%
20050831	1.32%	2.06%	2.51%
20050930	2.69%	2.48%	3.44%
20051031	-4.70%	-1.71%	-5.66%
20051130	2.32%	4.23%	4.19%
20051230	2.94%	2.10%	4.47%
20060131	2.30%	3.15%	6.05%
20060228	1.58%	1.21%	-1.99%
20060331	0.68%	1.81%	3.99%
20060428	2.53%	1.31%	0.89%
20060531	-1.95%	-1.01%	-3.53%
20060630	-2.42%	-1.83%	-0.82%
20060731	0.25%	3.25%	1.82%
20060831	2.51%	2.80%	2.26%
20060929	-0.79%	0.07%	-2.24%
20061031	2.69%	2.78%	5.10%
20061130	0.18%	3.62%	3.31%
20061229	2.72%	2.78%	1.50%
20070131	2.50%	-0.64%	1.11%
20070228	0.30%	0.16%	0.27%
20070330	0.77%	2.54%	1.19%
20070430	2.33%	4.08%	2.07%
20070531	2.57%	4.23%	4.98%
20070629	-0.75%	-2.57%	-0.78%
20070731	-0.95%	0.36%	-0.08%
20070831	-1.82%	-1.27%	-1.29%
20070928	0.76%	0.95%	3.46%
20071031	0.94%	4.33%	3.87%
20071130	-5.62%	-6.52%	-6.07%
20071231	0.71%	1.20%	1.36%
20080131	-6.68%	-5.30%	-4.76%
20080229	-0.85%	0.24%	3.47%
20080331	1.05%	1.25%	-1.42%
20080430	-0.56%	2.78%	4.60%
20080530	2.03%	2.10%	5.79%
20080630	-3.65%	-4.08%	-1.33%
20080731	-2.11%	0.78%	-5.86%
20080829	2.64%	4.64%	1.53%
20080930	-8.40%	-5.73%	-14.40%
20081031	-7.55%	-4.69%	-16.75%
20081128	-6.64%	-7.13%	-4.75%
20081231	4.75%	-0.93%	-2.48%
20090130	1.59%	-0.43%	-2.96%
20090227	-5.68%	-5.08%	-6.35%
20090331	0.32%	1.31%	7.87%
20090430	1.33%	4.43%	7.27%
20090529	2.68%	5.00%	11.46%

20090630	1.71%	2.92%	0.37%
20090731	0.38%	1.32%	4.26%
20090831	1.54%	3.96%	1.09%
20090930	1.74%	1.16%	5.15%
20091030	-0.65%	0.06%	-4.05%
20091130	4.42%	5.24%	5.15%
20091231	6.06%	6.34%	3.16%
20100129	-1.32%	-2.11%	-5.35%
20100226	5.00%	3.77%	4.97%
20100331	2.44%	4.88%	3.81%
20100430	-0.01%	0.35%	1.67%
20100528	-1.89%	-1.72%	-4.21%
20100630	-0.67%	-0.72%	-2.93%
20100730	4.09%	5.53%	4.05%
20100831	3.68%	1.20%	1.89%
20100930	3.76%	4.56%	4.08%
20101029	3.88%	1.50%	2.77%
20101130	-0.15%	-0.18%	2.37%
20101231	1.46%	3.23%	4.12%
20110131	1.03%	0.59%	0.99%
20110228	1.07%	2.23%	4.43%

Out-of-sample data: Canada

Period	Back test #8	S&P TSX
20110331	**1.17%**	**0.12%**
20110429	**-0.25%**	**-1.02%**
20110531	**2.54%**	**-0.88%**
20110630	**-1.57%**	**-3.28%**
20110729	**-1.70%**	**-2.51%**
20110831	**0.58%**	**-1.21%**
20110930	**-1.88%**	**-8.54%**
20111031	**1.70%**	**5.62%**
20111130	**0.94%**	**-0.22%**
20111230	**1.87%**	**-1.69%**
20120131	**0.55%**	**4.37%**
20120229	**1.30%**	**1.68%**

Global strategy back test # 7 In-sample data

Period	#7	MSCI WORLD
19990226	-1.19%	-2.65%
19990331	6.88%	4.18%
19990430	2.28%	3.96%
19990531	-1.32%	-3.64%
19990630	-1.09%	4.68%
19990730	-0.37%	-0.29%
19990831	-1.49%	-0.16%
19990930	-0.57%	-0.96%
19991029	-1.92%	5.21%
19991130	1.14%	2.83%
19991231	-3.07%	8.11%
20000131	-2.79%	-5.72%
20000229	4.41%	0.28%
20000331	-0.44%	6.92%
20000428	1.86%	-4.22%
20000531	1.56%	-2.52%
20000630	0.34%	3.38%
20000731	2.05%	-2.80%
20000831	1.18%	3.27%
20000929	-1.41%	-5.31%
20001031	1.27%	-1.66%
20001130	7.55%	-6.06%
20001229	-1.68%	1.63%
20010131	0.17%	1.94%
20010228	-1.38%	-8.44%
20010330	5.08%	-6.55%
20010430	0.96%	7.42%
20010531	0.99%	-1.24%
20010629	0.41%	-3.12%
20010731	2.10%	-1.32%
20010831	-5.10%	-4.78%
20010928	0.41%	-8.80%
20011031	1.85%	1.93%
20011130	0.90%	5.93%
20011231	-0.16%	0.64%
20020131	2.30%	-3.02%
20020228	4.45%	-0.85%
20020329	2.92%	4.44%
20020430	4.28%	-3.36%
20020531	-1.53%	0.23%
20020628	-5.11%	-6.05%
20020731	2.64%	-8.42%

20020830	-3.97%	0.21%
20020930	1.14%	-10.98%
20021031	-0.87%	7.40%
20021129	4.35%	5.41%
20021231	1.23%	-4.83%
20030131	1.30%	-3.02%
20030228	0.28%	-1.71%
20030331	4.77%	-0.27%
20030430	7.58%	8.93%
20030530	2.25%	5.76%
20030630	-0.18%	1.77%
20030731	2.24%	2.05%
20030829	3.92%	2.18%
20030930	5.33%	0.63%
20031031	3.42%	5.95%
20031128	6.54%	1.55%
20031231	2.50%	6.30%
20040130	3.22%	1.63%
20040227	-0.88%	1.71%
20040331	-2.71%	-0.62%
20040430	0.52%	-1.99%
20040531	2.29%	0.98%
20040630	-1.65%	2.10%
20040730	2.28%	-3.24%
20040831	2.08%	0.48%
20040930	2.01%	1.92%
20041029	6.83%	2.47%
20041130	4.37%	5.30%
20041231	-0.88%	3.85%
20050131	1.71%	-2.23%
20050228	-0.67%	3.21%
20050331	0.05%	-1.90%
20050429	0.99%	-2.11%
20050531	1.23%	1.85%
20050630	2.29%	0.91%
20050729	0.76%	3.52%
20050831	0.70%	0.80%
20050930	-3.82%	2.63%
20051031	1.61%	-2.41%
20051130	1.54%	3.39%
20051230	3.18%	2.24%
20060131	2.15%	4.48%
20060228	2.20%	-0.11%
20060331	3.99%	2.24%
20060428	-1.74%	3.09%
20060531	1.55%	-3.33%
20060630	2.78%	0.01%

20060731	3.07%	0.65%
20060831	1.17%	2.65%
20060929	3.51%	1.22%
20061031	3.84%	3.69%
20061130	3.37%	2.50%
20061229	0.07%	2.06%
20070131	-0.27%	1.20%
20070228	2.27%	-0.48%
20070330	4.49%	1.87%
20070430	2.42%	4.47%
20070531	-1.36%	2.90%
20070629	-2.34%	-0.74%
20070731	0.80%	-2.19%
20070831	4.89%	-0.03%
20070928	3.63%	4.79%
20071031	-1.25%	3.09%
20071130	-0.03%	-4.04%
20071231	-3.71%	-1.26%
20080131	0.82%	-7.62%
20080229	-1.31%	-0.53%
20080331	0.94%	-0.91%
20080430	2.25%	5.34%
20080530	-4.88%	1.65%
20080630	0.66%	-7.94%
20080731	0.96%	-2.42%
20080829	-3.39%	-1.36%
20080930	-9.70%	-11.85%
20081031	-1.11%	-18.93%
20081128	4.22%	-6.40%
20081231	-3.11%	3.26%
20090130	-9.11%	-8.73%
20090227	0.96%	-10.17%
20090331	4.45%	7.60%
20090430	6.08%	11.32%
20090529	3.55%	9.19%
20090630	5.45%	-0.41%
20090731	1.70%	8.50%
20090831	3.35%	4.17%
20090930	-1.32%	4.02%
20091030	3.11%	-1.76%
20091130	2.26%	4.14%
20091231	-1.47%	1.83%
20100129	2.17%	-4.11%
20100226	3.89%	1.45%
20100331	0.20%	6.25%
20100430	-5.49%	0.07%
20100531	1.66%	-9.48%

20100630	3.97%	-3.39%
20100730	1.70%	8.13%
20100831	4.01%	-3.69%
20100930	2.15%	9.36%
20101029	-2.40%	3.75%
20101130	3.96%	-2.11%
20101231	-0.26%	7.39%
20110131	3.28%	2.28%
20110228	-0.67%	3.55%
20110331	3.67%	-0.94%
20110429	2.02%	4.31%
20110531	-1.42%	-1.97%

Out-of-sample data for global strategy # 7

Period	Back test #7	MSCI WORLD
20110630	**-0.02%**	**-1.58%**
20110729	**0.19%**	**-1.81%**
20110831	**-2.79%**	**-7.05%**
20110930	**1.68%**	**-8.64%**
20111031	**-1.42%**	**10.34%**
20111130	**3.35%**	**-2.44%**
20111230	**-0.26%**	**-0.06%**
20120131	**1.97%**	**5.02%**
20120229	**0.37%**	**4.88%**

Appendix 2: Axioma risk models and optimization process

Throughout our study and back tests in this paper, we reply on risk modeling and portfolio optimization process provided by Axioma Inc. to create our minvar/low volatility high yield portfolios. Therefore, we would like to spend some time to describe and explain in detail how their risk models work, as well as the portfolio optimizer utilized during the strategy back tests with a monthly frequency. This section is organized as follows: 1. Portfolio risk management and Axioma's multi-factor risk models with factsheets. 2. Axioma portfolio optimization. 3. Discussion on advantages and limitation of applying a third party risk modeling and optimization techniques.

1. Portfolio risk management and Axioma's multi-factor risk models:

In order to construct and understand a portfolio by targeting either expected return or risk (standard deviation), one needs to firstly attempt to accurately estimate assets return covariance matrix, i.e. understand how each asset behaves from a risk-return trade off perspective, as well as how assets interact with each other. These are the input variables required to calculate expected portfolio returns and volatilities according to modern portfolio theory. One simple way to do this is to construct the covariance matrix directly by using observed historical return and risk data. Of course, this is also not the best solution as any of the following issues could cause significant estimation error: data mining, spurious relationships between assets and insufficient degree of freedom. (There are too many relationships to estimate among assets and not enough assets and observations to do so).

Therefore, a better approach is to identify "common factors" in the market place that drive asset returns and correlations so that a multi-factor risk model can be built to only estimate a limited number of parameters to explain and forecast risks. There are primarily two categories of these multi-factor risk models provided by Axioma Inc.: Fundamental model and statistical model. Fundamental risk model approach these common factors from the following three aspects: style, such as value, growth, size, etc. , country and industry classification factors and macro-economic factors such as GDP, equity market returns and Inflation etc. The estimation process is completed by conducting the following multi-variable regression analysis using historical data sets:

An asset's return is decomposed into a portion driven by these factors (common factor return) and a residual component, producing the following model at time t in matrix form:

$$r = Bf + u$$

where r is the vector of asset returns at time t, f the vector of common factor returns, and u, the set of asset specific returns. B is the n by m exposure matrix. Its elements denote each asset's exposure to a particular factor. For example, if one of the potential factors in the model is value and the model builder defines value as Price/Book Value (one can try many different types of value factors or combination of them to determine which one does the best as a common driver of returns and variance), each assets in the universe will have a normalized exposure to this factor "value", regressing the returns of all assets at time t on their exposures to value would generate a factor return for the value factor which reside in vector f. A multi factor regression will generate all the factor returns for all the factors considered and screen out the ones that consistently and significantly drive the asset return series over time, the left over effect will be in u, which denoted as the residual, asset specific returns in each assets that are not being captured by any of the common factors in f. The ultimate goal is to produce a vector of u that cannot be explained by any other common factors anymore. Expanding into other categories of common factors, the chosen fundamental risk model will have set of style factors, country/industry factors (no country factor if it is a risk model for only one country, such as the Canadian risk model) and macro-economic factors. As soon as the final model and factors are determined, a construction of an asset return covariance matrix is possible by completing the following calculation:

$$\text{Variance (r)} = \text{Variance (Bf + u)}$$
$$\text{Or}$$
$$Q = B\sum BT + \Delta 2$$

where Q is the m by m factor covariance matrix and Δ2 is the diagonal matrix of specific variances. In essence, the multi-factor model is a dimension reduction tool, simplifying the problem of calculating an n by n asset returns covariance matrix into calculating the variances and covariances of a much smaller number of factors, and n specific variances. Interested readers may wish to consider Grinold and Kahn (1995) or Zangari (2003) for a full exposition on factor risk models and their applications.

The statistical factors risk models are built in a similar fashion as the fundamental risk models, the only differences are: Instead of using a multiple variable regression approach, a principle component analysis (PCA) is used to identify common statistical factors (without fundamental meanings like value or growth, they are just "factors" that explains and drive volatilities and returns), this is somewhat similar to a step-wise regression approach to derive a uncorrelated residual return.

Please see the following fact sheet for detailed factor and estimation information on Axioma risk models:

Model Overview (Canada)

Asset Coverage	As of 2012, the model covers over 1,300 securities listed on the Toronto Stock Exchange, including Income Trusts and REITs.
Estimation Universe	Dynamic selection criteria are employed to identify TSX stocks with sufficient size and market liquidity. Common stocks, REITs, and Income Trusts are all eligible for membership. Throughout the model history, the estimation universe amounts to roughly 400 stocks on average.
Model Variants (4)	Medium- and short-horizon, fundamental and statistical factor models available. Model History Daily history from January 1999 onwards.
Forecast Horizon	Medium-horizon model: 3-6 months. Short-horizon model: 1-2 months.
Estimation Frequency	Factor exposures and covariances, asset specific risks estimated daily.

Fundamental Factor Model

Style Factors (9)

Growth	Plowback times return-on-equity
Leverage	Total debt to market capitalization

Liquidity	1 month average daily volume over market capitalization
Market Sensitivity	6 month daily beta
Medium-Term Momentum	Cumulative return over past year excluding most recent month
Short-Term Momentum	Cumulative return over past month
Size	Natural logarithm of market capitalization
Volatility	3 month average of absolute return over cross-sectional standard deviation
Value	Book-to-price

Industry Factors (21) GICS-based industry classification with 0/1 assignments.

Returns Model Uses style and industry factors to model local excess returns.

Returns History Medium-horizon model: 4 years of daily returns for factor correlations. 2years of daily returns for factor volatilities.

Short-horizon model: 2 years of daily returns for factor correlations, 2 years of daily returns for factor volatilities.

Estimation Robust linear regression using Huber weight function and square-root market capitalization weights.

Statistical Factor Model

Factor Structure 15 statistical factors.

Estimation 2-Pass Asymptotic Principal Components factor analysis with residual variance adjusted returns.

Returns History 1 year of daily asset returns are used.

Factor Volatilities / Covariances

Estimation Covariance of exponentially-weighted daily factor returns.

Half-life Parameters Medium-horizon model: 125 days for variances, 250 days for correlations. Short-horizon model: 60 days for variances, 125 days for correlations.

Autocorrelation Newey-West adjustment accounting for 1 day of autocorrelation is used in both fundamental and statistical factor models.

Adjustments Axioma's proprietary Dynamic Volatility Adjustment (DVA) procedure is used to analyze trends in factor returns dispersion and adjust risk estimation accordingly, to allow for heightened responsiveness in risk forecasts and adaptability to the prevailing volatility regime.

2. Axioma portfolio optimization process:

Once a risk model is built and chosen, one can then proceed to reply on it to create an "optimized" portfolio based on different types of portfolio objectives given the risk and return profiles are now properly modelled and hopefully accurately forecasted. Detail of the portfolio optimization process is described in section 1 of the paper. Here, we would like to introduce the unique features of the Axioma portfolio optimizer:

Axioma Portfolio uses Second Order Cone Programming (SOCP), a state-of-the-art algorithm capable of solving complex optimization problems exactly and efficiently. It provides the following features while most of the simpler optimizers do not:

- Risk as an explicit constraint, allowing one to target one or more risk parameters while solving the optimization problem. This is particularly useful while looking at a minvar or risk oriented strategy.

- Using more than one risk model in the same strategy.

- Using more than one benchmark or model portfolio in the same strategy.

- Diagnose infeasibilities quickly and easily:

The most common frustration with optimizers and back testing is an "infeasible" result. This occurs when two or more constraints in the strategy conflict. Axioma Portfolio's constraint hierarchy provides a comprehensive approach to dealing with infeasibilities. The user can provide a priority for each constraint and the optimizer will provide a solution that is "as close" to satisfying all the constraints as possible. If a constraint is violated, the constraint with the lowest possible priority is always chosen. This feature is particularly valuable for back testing. It is extremely common to run into infeasibilities over the course of a back test. The Constraint Hierarchy provides rules the optimizer can use to construct a reasonable solution and proceed with the back test.

3. Advantages and limitations of utilizing a third-party commercial risk modeling and optimization techniques.

Advantages:

- Third party providers specialize in building risk models and optimization engines, users can effectively focus more on their main tasks of building portfolio strategies and not worry about building and updating risk models.

- The technology and support from the vendors will provide users with more efficiency and leverage to quickly and effectively create back test facilities and strategy simulations while proprietary process usually takes a lot more resources to build and maintain.

- Market leaders in the risk modeling industry can accurately and dynamically capture the change in risk forecasting, effectively increase market efficiency for all users (No one needs to be out there by themselves when trying to understand risk independently)

Limitations:

- Because these products are commercial, users are unable to customize as much comparing to building their own models and systems, as a result, some conclusions are model / provider specific and may experience variance while switched to a different model/environment. Luckily, most of the common risk factors such as value, size, etc. have converged into very consistent definitions with high correlations across different vendors.

- Data set utilized by providers is also a source of limitation. For example, the estimation universe and type of data, as well as data frequency are set and if user has different preferences on these, it would be hard to apply. However, the new trend of risk modeling these days is to provide users with platforms to custom almost anything the user wants to create proprietary models in a commercial technology environment.